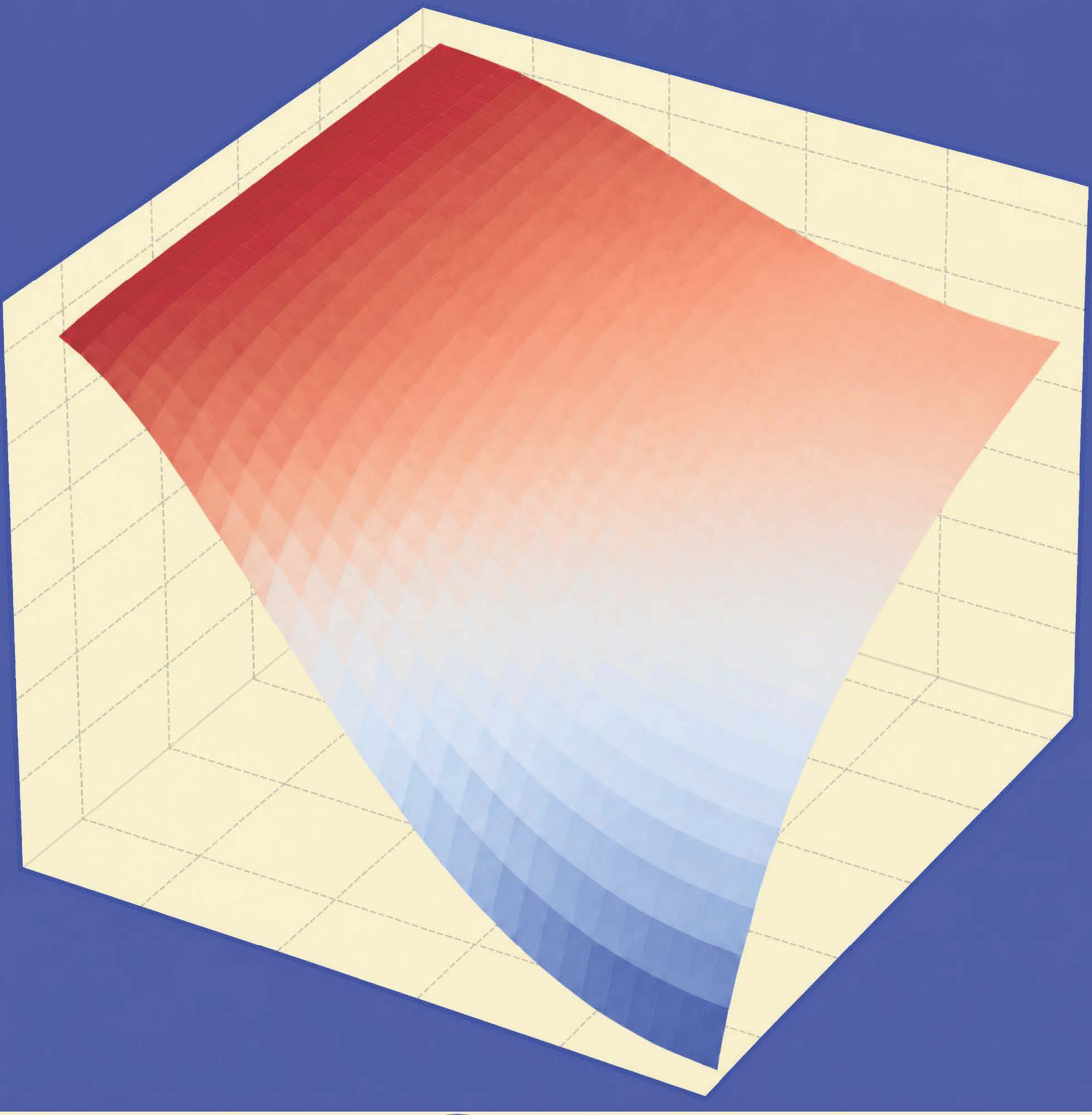
Vasile - Aurel CĂUŞ

Elements of q-calculus



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Vasile - Aurel CĂUȘ

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Author: Vasile - Aurel CĂUȘ

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Chapter 1

Introduction

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1.1 Methodological Framework

The techniques used throughout the book combine classical methods from geometric function theory with modern operator-based and q-analytic approaches. The principal tools include:

- Differential subordination and superordination, adapted to q-difference and symmetric q-difference operators.
- Hadamard convolution techniques, used to encode operator effects through coefficient modifications.
- Coefficient comparison principles leading to bi-directional inequalities and sharp coefficient bounds.
- Iterated and higher-order operators, which create hierarchical families of analytic subclasses.
- Analytic properties of special functions, especially q-hypergeometric and q-Bessel functions.
- Symmetric quantum derivatives, which provide balanced deformation mechanisms with improved geometric stability.

Together, these tools form a unified analytical strategy that governs the entirety of the research program.

1.2 Overview of the Main Contributions

The book introduces several novel operator-induced subclasses of analytic and meromorphic functions and provides sharp analytic characterizations for them. The major contributions can be found in [8, 12, 13, 11, 10, 7, 6, 5, 4] and may be summarized as follows:

(1) q-Difference and Bessel-type operator classes.

New families of analytic functions defined through generalized q -difference operators and q-Bessel-type transforms are introduced. Their geometric properties — starlikeness, convexity, growth, distortion — are obtained using subordination methods and coefficient inequalities.

(2) Janowski-type subclasses generated by higher-order q-derivatives.

Using iterated q-differential operators, the book defines q-Janowski classes that generalize classical Janowski theory. Sharp coefficient bounds, subordination criteria, and radii results are derived.

(3) Operator-induced classes generated by symmetric q-differential operators.

The symmetric q-derivative is used to define new subclasses of analytic and multivalent functions with enhanced geometric properties. Symmetry significantly improves the behaviour of bounds, leading to sharper inequalities and more stable operator dynamics.

(4) Meromorphic p-valent Janowski-type classes defined by iterated symmetric q-differential operators.

A completely new class of meromorphic functions is introduced through an operator $\widetilde{\mathcal{L}}^{\beta}_{\tau,q}$. Sharp characterizing inequalities, extremal functions, coefficient estimates, growth and distortion theorems, and radii of starlikeness and convexity are established.

(5) Unified operator framework across analytic and meromorphic contexts.

Across all chapters, the book develops a common structural methodology, showing that various classical results have natural q-analogues and symmetric q-analogues.

1.3 Preliminaries and Analytical Background

This chapter is devoted to preliminaries required for the development of the results presented in the subsequent chapters.

We introduce the necessary function classes, operators, and tools from geometric function theory, together with elements of q-calculus, symmetric q-operators, and fractional q-calculus. All notions recalled here are well established and serve as a common framework for the analysis carried out in the following chapters.

Let $\mathcal{H}(\mathbb{U})$ denote the collection of complex-valued functions that are analytic on the open unit disk,

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We define $\mathcal{A}_p$ as the subset of $\mathcal{H}(\mathbb{U})$ consisting of functions that satisfy the normalization condition

$$\mathcal{A}_p = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = z^p + a_{p+1}z^{p+1} + \dots,\ z \in \mathbb{U}\}, \tag{1.1}$$

where $p \in \mathbb{N} = \{1, 2, 3 \dots\}$. In the special case $p = 1$, this family coincides with the classical class $\mathcal{A} = \mathcal{A}_1 = \{f \in \mathcal{H}(\mathbb{U}) : f(0) = 0, f'(0) = 1\}$, consisting of normalized analytic functions on the unit disk.

Each function $f \in \mathcal{A}$ admits the power series representation

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,\ z \in U. \tag{1.2}$$

Having introduced the general class $\mathcal{A}_p$, we next consider certain subclasses that will play a role in future developments.

Within the framework of $\mathcal{A}_p$, we now focus on a subclass determined by the signs of the coefficients.We denote by $\mathcal{T}_p$ the subclass of $\mathcal{A}_p$ consisting of all functions with negative coefficients, that is,

$$f(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p}|\, z^{j+p}. \tag{1.3}$$

Such classes arise naturally in extremal problems and in the study of growth and distortion properties for p-valent functions.

Next, we turn to meromorphic p-valent functions. Let $\mathcal{M}_p$ denote the class of all meromorphic functions f that are p-valent ($p \in \mathbb{N}^* = \{1, 2, 3....\}$) and analytic in the punctured unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ ([42, 20]). Each function in this class is assumed to satisfy the normalization condition:

$$f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} a_{j+p} z^{j+p},\ z \in \mathbb{U}^*. \tag{1.4}$$

Clearly, when $p = 1$, the class $\mathcal{M}_p$ reduces to the class $\mathcal{M}$ of univalent and meromorphic functions (see [20, 18]).

Since subordination relations are often expressed through Schwarz functions, we briefly recall their definition and a key lemma.

Let Ω denote the class of Schwarz functions $\Phi(z)$ of the form

$$\Phi(z) = \sum_{j=1}^{\infty} c_j z^j, z \in \mathbb{U},$$

which are analytic in the unit disk and satisfying

$$\Phi(0) = 0 \text{ and } |\Phi(z)| < 1,\ z \in \mathbb{U}. \tag{1.5}$$

Lemma 1.3.1. *(Schwarz lemma) If $\Phi(z) \in \Omega$, then for all $z \in \mathbb{U}$,*

$$|\Phi(z)| \leq |z|$$

and for the first coefficient, we have

$$|c_1| \leq 1.$$

One of the central tools used throughout this work is differential subordination and its dual concept, differential superordination.

Differential subordination is a powerfull technique used to establish the conditions under which an analytic function is subordinate to another.

$$w(0) = 0,\ |w(z)| < 1 \text{ and } f(z) = g(w(z)).$$

If g is also a univalent in U, then the relation $f \prec g$ is equivalent to the condition:

$$f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Now, let ψ be a given mapping from $\mathbb{C}^3 \times U$ to $\mathbb{C}$ and let h be an injective function in U. A function, s, holomorphic in U is said to satisfy a second-order differential subordination if

$$\psi(s(z), zs'(z), z^2 s''(z); z) \prec h(z). \tag{1.6}$$

In this setting, a function, u, univalent in U, is called a dominant of all the solutions of (1.6) if

$$s \prec u,$$

for all s satisfying (1.6). Furthermore, if u is the smallest such dominant in a subordination sense — meaning that for every other dominant, v, of (1.6), we have

$$u(z) \prec v(z),$$

then u is referred to as the optimal dominant. Except for a rotational transformation of the unit disk U, the leading dominant remains singular, illustrating the inherent rotational symmetry present in the class of Schwarz functions.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be a given function, and let h be an analytic function defined in U. Suppose that s is a holomorphic mapping in U such that both s and $\psi(s(z), zs'(z), z^2 s''(z); z)$ are univalent in U. If the differential superordination condition

$$h(z) \prec \psi(s(z), zs'(z), z^2 s''(z); z) \tag{1.7}$$

is satisfied, then the function s constitutes the fulfillment of the second-order differential inequality condition defined by (1.7) (see [29]). In this context, the function u, being analytic, is regarded as the subordinate entity under the differential superordination condition (1.7) if

$$u \prec s,$$

for all s satisfying (1.7). If u is the greatest such function in the sense of subordination — i.e., if

$$v(z) \prec u(z),$$

for every subordinant v of (1.7) then u is called the best subordinant. This function represents the maximal element (with respect to subordination) among all the admissible subordinants and is unique up to rotation in U.

In their foundational work, Miller and Mocanu [30] proposed that the implication below is valid under appropriate constraints on the functions h, u, and ψ:

$$h(z) \prec \psi(s(z), zs'(z), z^2 s''(z); z) \Rightarrow u(z) \prec s(z).$$

Under these conditions, the function u serves as the best subordinant for the given differential superordination problem. Under the established framework of differential subordination and superordination, one is often interested not only in identifying extremal functions like the best subordinant, but also in exploring structural operations that preserve or reveal the subordination properties. One such operation, which plays a central role in geometric function theory, is the Hadamard product (or convolution). This operation allows us to combine analytic functions in a coefficient-wise manner and examine how subordination relationships behave under such combinations.

Let us consider two functions that are holomorphic in U, given by

$$f(z) = z^p + \sum_{j=2}^{\infty} a_{j+p} z^{j+p} \text{ and } g(z) = z^p + \sum_{j=2}^{\infty} b_{j+p} z^{j+p}.$$

The function $(f * g)(z)$, representing the Hadamard (convolution) product of f and g, is expressed as

$$f(z) * g(z) = (f * g)(z) = z^p + \sum_{j=2}^{\infty} a_{j+p} b_{j+p} z^{j+p}.$$

Below, we present the definition of the Al-Oboudi operator, which will play a key role in obtaining our forthcoming results.

Definition 1.3.2 (Al Oboudi [34]). *Let f be an element of the class $\mathcal{A}$, λ a non-negative real number, and m a natural number; the operator D_λ^m is described as $D_\lambda^m : \mathcal{A} \to \mathcal{A}$:*

$$\begin{aligned} D_\lambda^0 f(z) &= f(z), \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \\ &\dots \\ D_\lambda^m f(z) &= (1-\lambda) D_\lambda^{m-1} f(z) + \lambda z (D_\lambda^m f(z))' = D_\lambda \left(D_\lambda^{m-1} f(z)\right), \text{ for } z \in U. \end{aligned}$$

Remark 1.3.3 (Al Oboudi [34]). *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*

$$D_\lambda^m f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m a_j z^j, \text{ for } z \in U. \quad (1.8)$$

Remark 1.3.4. *The earlier formulation reduces to the Sălăgean differential operator upon setting $\lambda = 1$ [27].*

Definition 1.3.5 ([29]). *Denote by Q the set of all the functions, f, that are analytic and injective on $\overline{U} \backslash F(f)$, where $F(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash F(f)$.*

Lemma 1.3.6 ([29]). *Assume that the function u is one-to-one in U and that θ and ϕ are holomorphic in a domain, D, which contains the image of $u(U)$. Further suppose that*

$\phi(w) \neq 0$ for every $w \in u(U)$. Define the auxiliary functions $Q(z) = zu'(z)\phi(u(z))$ and $h(z) = \theta(u(z)) + Q(z)$. Let the following conditions be satisfied:

1. Q is starlike and injective in U;

2. $\mathrm{Re}\left(\frac{sh'(z)}{Q(z)}\right) > 0$, for all $z \in U$.

Now, consider a function, s, that is analytic in U, such that

$$\begin{aligned} s(0) &= u(0), \\ s(U) &\subseteq D, \end{aligned}$$

and

$$\theta(s(z)) + zs'(z)\phi(s(z)) \prec \theta(u(z)) + zu'(z)\phi(u(z)).$$

Then it follows that

$$s(z) \prec u(z),$$

and u is the extremal function in this subordination framework.

Lemma 1.3.7 ([9])**.** *Let u be a convex and injective function within U, and suppose that v and ϕ are holomorphic in a region, D, that includes the image of $u(U)$. Assume that the following conditions hold:*

1. $\mathrm{Re}\left(\frac{v'(u(z))}{\phi(u(z))}\right) > 0$, for $z \in U$;

2. $\psi(z) = zu'(z)\phi(u(z))$ is univalent and starlike in U.

Now consider a function, $s(z)$, satisfying

$$\begin{aligned} &s(z) \in \mathcal{H}[u(0), 1] \cap Q, \\ &s(U) \subseteq D, \\ &v(s(z)) + zs'(z)\phi(s(z)) \text{ is injective in } U, \end{aligned}$$

and

$$v(u(z)) + zu'(z)\phi(u(z)) \prec v(s(z)) + zs'(z)\phi(s(z)).$$

Then it follows that

$$u(z) \prec s(z),$$

and u is the best subordinant.

Before turning to specific function classes, we also recall a family of functions that frequently appears in subordination and univalence criteria.

We define by $\mathcal{P}$ the family of holomorphic functions $\theta(z)$ defined in U, whose range lies entirely in the right half plane:

$$Re(\theta(z)) > 0, \text{ for all } z \in U,$$

and wich are normalized by $\theta(0) = 1$. These mappings serve as a cornerstone in geometric function theory, as they frequently arise in the investigation of univalent and starlike function classes. Numerous foundational results and deep structural properties have been developed with respect to this class. Each function $\theta \in \mathcal{P}$ admits a development in series expressed as

$$\theta(z) = 1 + \sum_{j=1}^{\infty} w_j z^j, \ z \in U,$$

where the coefficients w_j are complex-valued and the series converges for all $z \in U$. These functions are commonly referred to as Carathéodory functions.

We denote by S be the class all functions in $\mathcal{A}$ which are univalent in U. Denote by S^* the subclass of functions $f(z) \in S$ that are starlike with the respect to the origin. Anaytically, it is well-known that $f(z) \in S^*$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U.$$

The class $S^*(\alpha)$ of starlike functions of order α consist of thodse functions $f(z) \in \mathcal{A}$ that satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U,\ 0 \le \alpha < 1.$$

Beyond starlike and Janowski-type classes, further generalizations have been proposed in the literature. We briefly recall several of them, as they will be relevant for our applications.

We recall that Janowski introduced a generalization of the class $\mathcal{S}^*$ as follows:

Definition 1.3.8. *[24] We say that $f \in \mathcal{A}$ lies in the subclass $\mathcal{S}^*(A, B)$ if:*

$$\operatorname{Re}\left(\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}\right) \ge 0,\ -1 \le B < A \le 1.$$

Equivalently, this condition can be written as a subordination relation of the form

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\tau)z}{1-z}, z \in U,\ 0 \le \tau < 1.$$

for a suitable choice of parameters.

Let K, C denote the usual subclass of $\mathcal{A}$ whose members are close-to-convex, convex in the open unit disk U. Gao and Zhou [17] introduced the class K_s of analytic functions, which is a subclass of the class C. We say that a function $f \in \mathcal{A}$ is in the class K_s, if there exists a starlike function $g \in S^*\left(\frac{1}{2}\right)$ such that

$$\operatorname{Re}\left\{-\frac{z^2 f'(z)}{g(z)\,g(-z)}\right\} > 0,\ z \in U.$$

In [19], Goyal, Singh and Bulboacă defined and studied a subclass of analytic functions related to starlike functions. If $f \in \mathcal{A}$, we say that $f \in K_s(A, B; u, v)$ if there exists a function $g \in S^*\left(\frac{1}{2}\right)$ such that

$$\frac{uvz^2 f'(z)}{g(uz)\,g(vz)} \prec \frac{1+Az}{1+Bz},\ z \in U.$$

$(-1 \le B < A \le 1, u, v \in \mathbb{C}^*, |u| \le 1, |v| \le 1)$.

More recently, J.K. Prajapat [36] introduced and studied a new subclass of analytic functions $\chi_t(\gamma)$. A function $f \in \mathcal{A}$ is said to be in the class $\chi_t(\gamma)$, $(0 \le \gamma < 1, |t| \le 1, t \ne 0)$, if there exists a function $g \in S^*\left(\frac{1}{2}\right)$ for which

$$\operatorname{Re}\left\{\frac{tz^2 f'(z)}{g(z)\,g(tz)}\right\} > \gamma,\ z \in U.$$

A function $f \in \mathcal{M}_p$ is said to belong to the class $\mathcal{SM}_p^*(\alpha)$, where $0 \le \alpha < p$, of meromorphic p-valent starlike functions of order α (see [40, 39]), if:

$$f(z) \in \mathcal{SM}_p^*(\alpha) \iff \operatorname{Re}\left\{\frac{zf'(z)}{pf(z)}\right\} < -\alpha, \quad z \in \mathbb{U}^*,\ 0 \le \alpha < p. \tag{1.9}$$

In particular, for $\alpha = 0$, the class $\mathcal{SM}_p^*(0)$ coincides with $\mathcal{SM}_p^*$, the class of meromorphic p-valent starlike functions.

The class $\mathcal{CM}_p(\alpha)$, $0 \le \alpha < p$, consists of meromorphic p-valent convex functions of order α (see [18]). A function $f \in \mathcal{M}_p$ is said to belong to $\mathcal{CM}_p(\alpha)$ if for $z \in \mathbb{U}^*$:

$$\operatorname{Re}\left\{\frac{\left(zf'(z)\right)'}{pf'(z)}\right\} < -\alpha, \quad z \in \mathbb{U}^*,\ 0 \le \alpha < p. \tag{1.10}$$

Similarly, we denote by $\mathcal{KM}_p^*(\alpha)$ the class of meromorphic p-valent close-to-convex functions of order α (see [39]), defined by:

$$f(z) \in \mathcal{KM}_p^*(\alpha) \iff \operatorname{Re}\left\{\frac{zf'(z)}{pe(z)}\right\} < -\alpha, \quad z \in \mathbb{U}^*,\ 0 \le \alpha < p, \tag{1.11}$$

where $e(z) \in \mathcal{SM}_p^*$, for $z \in \mathbb{U}^*$. Since several results in this work involve q-deformations of classical function classes, we next summarize the necessary background on q-calculus.

We briefly recall some notations and concepts of q-calculus. The theory of q-analogues or q-extensions of classical formulas and functions is based on the observation that

$$\lim_{q\to 1}\frac{1-q^j}{1-q} = j, q \in (0,1), j \in \mathbb{N}, \tag{1.12}$$

The quantity

$$[j]_q = \frac{1-q^j}{1-q}$$

is called the basic number (or q-number). The q-factorial $[j]_q!$ is defined by

$$[j]_q! = \begin{cases} [j]_q \cdot [j-1]_q \cdots [1]_q, \text{for } j = 1, 2, ...; \\ 1, \text{for } j = 0. \end{cases} \tag{1.13}$$

As $q \to 1^-$, $[j]_q \to j$, and this is the bookmark of a q-analogue: the limit as $q \to 1$ recovers the classical object.

In [23] and [22] Jackson introduced the q-difference operator $(D_q f)(z)$ acting on functions $f(z) \in \mathcal{A}$ defined as follows:

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)},\ z \ne 0,\ 0 < q < 1; (D_q f)(z)\,|_{z=0} = f'(0). \tag{1.14}$$

One can see that $(D_q f)(z) \to f'(z)$ as $q \to 1^-$.

The q-difference operator plays an important role in the theory of hypergeometric series and quantum phisics (see [2], [16]). Therefore, for a function $f(z) = z^j$ the q-derivative is given by

$$D_q f(z) = D_q\left(z^j\right) = \frac{1-q^j}{1-q} \cdot z^{j-1} = [j]_q\, z^{j-1}, \tag{1.15}$$

then $\lim\limits_{q\to 1} D_q f(z) = \lim\limits_{q\to 1} [j]_q\, z^{j-1} = jz^{j-1} = f'(z)$, where $f'(z)$ is the ordinary derivative.

From (1.14) we have

$$D_q f(z) = 1 + \sum_{j=2}^{\infty} \frac{1-q^j}{1-q} a_j z^j,\ z \ne 0.$$

For the function $f \in \mathcal{M}_p, f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} a_{j+p} z^{j+p}$, $z \in \mathbb{U}^*$, the power series of $\widetilde{D}_q f$ is

$$z\widetilde{D}_q f(z) = -\frac{\widetilde{[p]_q}}{z^p} + \sum_{j=1}^{\infty} \widetilde{[j+p]_q} a_{j+p} z^{j+p}. \tag{1.16}$$

Under the hypothesis of the definition of q-derivates operator, for $f, g \in \mathcal{A}$ we have the folowing rules:

$$D_q((af(z)) \pm bg(z)) = aD_q f(z) \pm bD_q g(z), \ a, b \in \mathbb{C},$$

$$D_q(f(z) g(z)) = g(z) D_q f(z) + f(qz) D_q g(z),$$

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z) D_q f(z) - f(z) D_q g(z)}{g(z) g(qz)}, \ g(z) g(qz) \neq 0.$$

The q-integral of f over $[0, z]$ is given by (see [23])

$$\int_0^z f(t) d_q t = z(1-q) \sum_{j=0}^{\infty} q^j f(q^j z),$$

provided the series converges.

In the limit $q \to 1^-$, this reduces to the usual integral

$$\lim_{q \to 1} \int_0^z f(t) d_q t = \int_0^z f(t) dt.$$

Higher-order q-derivatives are defined recursively by

$$D_q^{(0)} f(z) = f(z), \quad D_q^{(r)} f(z) = D_q\left(D_q^{(r-1)} f(z)\right).$$

Explicitly, the q-derivative of the function $f(z)$ of order r is given by

$$D_q^{(r)} f(z) = \frac{[p]_q!}{[p-r]_q!} z^{p-r} + \sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j+p-r]_q!} a_{j+p} z^{j+p-r}, \tag{1.17}$$

for $0 \leq r \leq p-1$ and $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

The definition of the q-Gamma function $\Gamma_q(x)$ is given by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{j=0}^{\infty} \frac{1-q^{j+1}}{1-q^{j+x}}, x > 0,$$

which satisfies the following fundamental properties:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x),$$

and

$$\Gamma_q(j+1) = [j]_q!,$$

where $j \in \mathbb{N}$ and the q-factorial $[j]_q!$ is defined in Equation (1.13).

We next recall some definitions related to fractional q-calculus operators for complex-valued functions $f(z)$.

Definition 1.3.9 ([37]). *The fractional q-integral of order $\alpha > 0$ for the function f is defined by*

$$(\mathcal{I}_{q,z}^{\alpha} f)(z) = D_{q,z}^{-\alpha} f(z) = \frac{1}{\Gamma_q(\alpha)} \int_0^z (z - qt)_{\alpha-1} f(t)\, d_q t,$$

with $f(z)$ analytic in a simply connected region of the complex plane that contains the origin. Here $(z - qt)_{\alpha-1}$ denotes the standard fractional q-shifted kernel used in fractional q-calculus, taken on the branch for which $\log(z - qt)$ is real whenever $z - qt > 0$. Throughout, we assume the principal branch and that the resulting integrals/series are absolutely convergent for $z \in \mathbb{U}$.

Definition 1.3.10 ([37]). *The fractional q-derivative of order α for the function f is defined by*

$$(\mathcal{D}_{q,z}^{\alpha} f)(z) = \mathcal{D}_{q,z} \mathcal{I}_{q,z}^{1-\alpha} f(z) = \frac{1}{\Gamma_q(1-\alpha)} D_{q,z} \int_0^z (z - qt)_{-\alpha} f(t)\, d_q t, 0 \leq \alpha < 1.$$

For each $j \in \mathbb{N}$, the expression $\widetilde{[j]}_q$, referred to as the symmetric q-analogue of a number, is introduced below:

$$\widetilde{[j]}_q = \frac{q^{-j} - q^j}{q^{-1} - q} \text{ with } \widetilde{[0]}_q = 0. \tag{1.18}$$

It is important to emphasize that the symmetric q-number differs from the classical (non-symmetric) q-number typically used within the framework of quantum harmonic oscillators under q-deformation [14]. This distinction is significant, especially in applications involving symmetry and operator algebra.

The symmetric q-shifted factorial, denoted by $\widetilde{[j]}_q!$, is defined recursively in analogy with the classical factorial, but using symmetric q-numbers:

$$\widetilde{[j]}_q! = \begin{cases} \widetilde{[j]}_q \cdot \widetilde{[j-1]}_q \cdot \widetilde{[j-2]}_q \cdot \cdot \widetilde{[1]}_q, \text{ for } j = 1, 2, ...; \\ 1, \text{ for } j = 0. \end{cases} \tag{1.19}$$

A notable limiting property is

$$\lim_{q \to 1^-} \widetilde{[j]}_q = j,$$

which shows that the symmetric q-factorial converges to the classical factorial in the standard limit$^-$.

We now define the symmetric q-derivative operator [25], denoted as $\widetilde{D}_q f(z)$, which acts on functions $f \in \mathcal{A}_p$. The operator is defined by

$$\widetilde{D}_q f(z) = \begin{cases} \frac{f(qz) - f(q^{-1}z)}{z(q - q^{-1})}, z \neq 0, q \neq 1, z \in U; \\ f'(0), \text{ for } z = 0. \end{cases} \tag{1.20}$$

As $q \to 1^-$, the operator converges to the classical derivative:

$$\lim_{q \to 1^-} \widetilde{D}_q f(z) = f'(z).$$

A direct computation confirms that the symmetric q-derivative of the monomial z^p yields

$$\widetilde{D}_q z^p = \widetilde{[p]}_q z^{p-1},$$

and for a general function, $f(z)$, its symmetric q-derivative has the following power series expansion:

$$z\widetilde{D}_q f(z) = \widetilde{[p]}_q z^p + \sum_{j=1}^{\infty} \widetilde{[j+p]}_q a_{j+p} z^{j+p}. \tag{1.21}$$

The operator $\widetilde{D}_q$ satisfies several essential identities [26], which mirror the properties of the classical derivative but reflect the underlying symmetric q-structure:

$$\begin{aligned}
\widetilde{D}_q (f(z) + g(z)) &= \widetilde{D}_q f(z) + \widetilde{D}_q g(z), \\
\widetilde{D}_q (f(z) g(z)) &= g\left(q^{-1}z\right) \widetilde{D}_q f(z) + f(qz) \widetilde{D}_q g(z) \\
&= g(qz) \widetilde{D}_q f(z) + f\left(q^{-1}z\right) \widetilde{D}_q g(z), \\
\widetilde{D}_q \left(\frac{f(z)}{g(z)}\right) &= \frac{f(qz) \widetilde{D}_q g(z) - g\left(q^{-1}z\right) \widetilde{D}_q f(z)}{g\left(q^{-1}z\right) g(qz)}, \\
\widetilde{D}_q f(z) &= \widetilde{D}_{q^2} f\left(q^{-1}z\right).
\end{aligned}$$

The difference operator helps us to generalize the class of starlike, convex, close-to-convex functions analitically. The q-analogues to the functions classes S^* and $\mathcal{K}$ are given as follows.

A function $f \in \mathcal{A}$ is said to belong to the class S_q^* of q-starlike functions if it satisfies

$$\left| \frac{z(D_q f)(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in U. \tag{1.22}$$

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{K}_q$ of q-close-to-convex functions if there exists a starlike function $g \in S^*$ such that

$$\left| \frac{zD_q f(z)}{g(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in U. \tag{1.23}$$

In this case we say that $f \in \mathcal{K}_q$ with respect to g. Some properties of functions in the class $\mathcal{K}_q$ were studied in [38] and [41]. In [35], Y. Polatoglu investigated basic characterization, growth and distortion theorems for the class S_q^*.

In [1], S. Agrawal and S.K.Sahoo introduced the class of q-starlike functions of order α, denoted by $S_q^*(\alpha)$.

A function $f \in \mathcal{A}$ is said to belong to the class $S_q^*(\alpha)$, for $0 \leq \alpha < 1$, if

$$\left| \frac{zD_q(f(z))}{f(z)} - \frac{1-\alpha q}{1-q} \right| \leq \frac{1-\alpha}{1-q}, z \in U.$$

In particular, when $\alpha = 0$, the class $S_q^*(\alpha)$ coincides with the class S_q^*, which was introduced by Ismail et.al.(see [21])

Let $\mathcal{S}_q(\beta, \theta)$ denote the subclass of $\mathcal{A}$, defined through analytic subordination for $0 \leq \beta < 1$ and a function $\theta \in \mathcal{P}$, as described in [3]:

$$\mathcal{S}_q(\beta, \theta) = \left\{ f \in \mathcal{A} : \frac{1}{1-\beta} \left(\frac{zD_q f(z)}{f(z)} - \beta \right) \prec \theta(z), \, z \in U \right\}.$$

Remark 1.3.11. *As q approaches 1 from the left, $\mathcal{S}_q(\theta)$ recovers the family $\mathcal{S}(\theta)$, as considered by [28].*

Remark 1.3.12. *As $q \to 1^-$, the q-analog class $\mathcal{S}_q(\beta, \theta)$ converges to the classical class $\mathcal{S}(\beta, \theta)$, which is defined through standard analytic subordination, as detailed in [15].*

Remark 1.3.13. *Furthermore, when $q \to 1^-$ and the function $\theta(z)$ is chosen as $\frac{1+z}{1-z}$, $\mathcal{S}_q(\theta)$ recovers the conventional starlike function class.*

Remark 1.3.14. *When the function $\theta(z)$ is taken to be $\frac{1+Az}{1+Bz}$, where the parameters satisfy $-1 \leq B < A \leq 1$, the corresponding class $\mathcal{S}_q(\theta)$ becomes equivalent to the class $\mathcal{S}_q(A, B)$, which was studied in detail by Noor and collaborators, as documented in [33]. Furthermore, in the limiting case as q approaches 1^-, $\mathcal{S}_q(A, B)$ converges to $\mathcal{S}(A, B)$, previously analyzed in the work of Janowski, referenced [24].*

Remark 1.3.15. *If the function $\theta(z)$ is defined as $\frac{1}{1-qz}$, then the associated class $\mathcal{S}_q(\theta)$ is equivalent to the class considered by Noor in [31].*

Remark 1.3.16. *When $\theta(z) = \frac{1+z}{1-qz}$, the corresponding class $\mathcal{S}_q(\theta)$ coincides with the q-starlike function class $\mathcal{S}_q(1, -q)$, discussed in [43] and [32].*

Equivalently, membership in the family $\mathcal{S}_q(1, -q)$ of starlike mappings defined in terms of q-calculus in U (see [21]), is characterized by the condition:

$$\left| \frac{zD_q f(z)}{f(z)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

1.4 Concluding Remarks

This introductory chapter positions the book within the broader context of geometric and quantum calculus-based operator theory. The results presented in the following chapters demonstrate that the interaction between classical geometric function theory and q-analytic structures is not only natural but also remarkably fruitful. By introducing new operators, developing sharp inequalities, and extending classical subclasses into quantum-deformed settings, the book contributes to the deepening of the analytical foundations of modern geometric function theory.

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Chapter 2

q-Difference Operators and Geometric Properties of Analytic Function Classes

Contents

The interplay between q-calculus and geometric function theory has attracted considerable attention in recent years, mainly due to the rich family of analytic transformations generated by q-difference operators and q-special functions. These operators provide natural q-analogues of classical differential operators and lead to new subclasses of analytic and univalent functions.

In the first part of this chapter (see [4]), we introduce and investigate a generalized q-close-to-convex class $K_{t,q}$, defined through a structural condition involving the q-difference operator and an auxiliary starlike function. This framework allows us to derive subordination results, coefficient inequalities and Bieberbach–de Branges type estimates.

In the second part (see [5]) , we turn to operator-induced subclasses generated by generalized differential operators associated with normalized Jackson q-Bessel functions. These operators provide a complementary mechanism for constructing q-starlike functions of order α and for extending classical results to q-deformed settings.

Together, these two approaches illustrate how q-difference tools contribute to the development of geometric function theory and highlight the unifying role played by subordination techniques in both structural and operator-theoretic contexts.

2.1 The Generalized q-Difference Class $K_{t,q}$

In followings we give a generalization of the class introduced in [10], by using the q-difference operator.

Motivated by the recent developments in q-calculus and its applications to geometric function theory, we now introduce a generalized q-difference close-to-convex class, denoted by $K_{t,q}$. This class extends previously studied families and serves as a natural setting for deriving subordination and coefficient estimates.

Definition 2.1.1. *A function $f \in \mathcal{A}$ is said to be in the class $K_{t,q}$, ($|t| \leq 1$, $t \neq 0, q \in (0,1)$), if there exists a function $g \in S^*\left(\frac{1}{2}\right)$ for which*

$$\left|\frac{tz^2 D_q f(z)}{g(z)\,g(tz)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q},\ z \in U. \tag{2.1}$$

We say that $f \in K_{t,q}$ with respect to g.

We give an example of a function belonging to this class.

Example 2.1.2. *The function $f_0(z) = z + \frac{z^2}{1+q}$ belongs to the class $K_{t,q}$ with respect to $g_0(z) = z, |t| \leq 1,\ t \neq 0, q \in (0,1),\ z \in U$. Indeed, f_0 is analytic in U with $f_0(0) = 0 = f_0'(0) - 1$ and $g_0 \in S^*\left(\frac{1}{2}\right)$. We have*

$$\left|\frac{tz^2 D_q f(z)}{g(z)\,g(tz)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q},\ z \in U \iff |z(1-q) - q| \leq 1,\ z \in U$$

This means that $f_0 \in K_{t,q}$ with respect to $g_0(z) = z$.

The following result plays a crucial role in understanding the structure of the class $K_{t,q}$. It establishes that the auxiliary function G, defined in terms of a starlike function g, remains starlike. This allows us to replace the dependence on g by a single starlike majorant G.

Theorem 2.1.3. *Let $g \in S^*\left(\frac{1}{2}\right)$, defined by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ z \in U, \tag{2.2}$$

and let

$$G(z) = \frac{g(z)\,g(tz)}{tz} = z + \sum_{n=2}^{\infty} c_n z^n, \;\; z \in U, \tag{2.3}$$

where $c_n = \sum_{j=1}^{n} b_j b_{n-j+1} t^{j-1}$, *with* $b_1 = 1$, $|t| \le 1$, $t \ne 0$. *Then* $G(z) \in S^*$.

Proof. Since $g \in S^*\left(\frac{1}{2}\right)$ and from the definition of starlike functions, we get $\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \frac{1}{2}$, $z \in U$. We note that for $z \in U$, we have $|tz| \le |z| \le 1$. So, we obtain $\operatorname{Re}\left\{\frac{tzg'(tz)}{g(tz)}\right\} > \frac{1}{2}$.

Therefore, $\operatorname{Re}\left\{\frac{zG'(z)}{G(z)}\right\} = \operatorname{Re}\left\{\frac{zg'(z)}{g(z)} + \frac{tzg'(tz)}{g(tz)} - 1\right\} > \frac{1}{2} + \frac{1}{2} - 1 = 0$. This proves the conclusion of the theorem. □

This result shows that the condition defining $K_{t,q}$ can be reformulated in terms of the function G, which simplifies the subsequent analysis and leads naturally to subordination relations.

Remark 2.1.4. *According to Theorem 2.1.3, the relation (2.1) is equivalent to*

$$\left| \frac{zD_q f(z)}{G(z)} - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \tag{2.4}$$

thus it is obvious that $K_{t,q} \subset \mathcal{K}_q$.

In this section, we introduce a new class of analytic functions $K_{t,q}$ in the unit disk U by making systematic use of the previously defined q-difference operator. Within this unified framework, we established several basic properties of this class, including its relationship with the classical close-to-convex functions and with the class $K_{t,q}$ defined in terms of q-difference conditions.

The main tools employed were subordination, coefficient comparison, and Clunie's method. In particular, the subordination relation

$$\frac{tz^2 D_q f(z)}{g(z)\,g(tz)} \prec \frac{1+z}{1-qz},$$

allowed us to derive inclusion results, growth estimates, and sharp coefficient inequalities for functions in $K_{t,q}$.

We also obtained sufficient conditions for membership in this class in terms of the Taylor coefficients of f and of the associated starlike function g.

Furthermore, we established a Bieberbach–de Branges type theorem for the generalized class $K_{t,q}$, providing explicit upper bounds for the coefficients $|a_n|$ in terms of the parameters q and t. These results extend known coefficient estimates for close-to-convex and q-close-to-convex functions to the more general q-difference setting.

The analysis presented here indicates that classes such as $K_{t,q}$, defined in terms of q-difference operators and auxiliary starlike functions, form a natural bridge between classical geometric function theory and its q-deformed analogue. This perspective will be further developed in the second part of the section, where we study operator-induced subclasses associated with normalized q-Bessel functions.

2.2 Inclusion, Subordination, and Coefficient Estimates for $K_{t,q}$

Using the representation obtained above, we now provide a subordination characterization of the class $K_{t,q}$. Such characterizations are essential for deriving coefficient bounds and inclusion relations.

Theorem 2.2.1. *Let $f \in \mathcal{A}$ given by the power series representation*

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \ z \in U. \tag{2.5}$$

We have $f \in K_{t,q}$ with respect to the function $g \in S^*\left(\frac{1}{2}\right)$ if and only if

$$\frac{tz^2 D_q f(z)}{g(z)\, g(tz)} \prec \frac{1+z}{1-qz}, \ z \in U. \tag{2.6}$$

Proof. Let $f(z)$ be an element of $K_{t,q}$. We have

$$\left| \frac{tz^2 D_q f(z)}{g(z)\, g(tz)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \ q \in (0,1), |t| \leq 1, t \neq 0, z \in \mathbb{U},$$

which is equivalent with

$$\left| \frac{tz^2 D_q f(z)}{g(z)\, g(tz)} - M \right| \leq M, \quad M = \frac{1}{1-q}, \ M > 1, |t| \leq 1, t \neq 0, z \in U.$$

So, the function

$$\varphi(z) = \frac{1}{M} \frac{tz^2 D_q f(z)}{g(z)\, g(tz)} - 1$$

has modulus at most 1 in the unit disk U. Therefore,

$$\phi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)} = \frac{\frac{1}{M}\frac{tz^2 D_q f(z)}{g(z)g(tz)} - \left(\frac{1}{M} - 1\right)}{1 - \left(\frac{1}{M} - 1\right)\left(\frac{1}{M}\frac{tz^2 D_q f(z)}{g(z)g(tz)} - 1\right)} \tag{2.7}$$

satisfies the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$. By Schwarz Lemma we get

$$\phi(z) \leq z. \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\frac{tz^2 D_q f(z)}{g(z)\, g(tz)} = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{M}\right)\phi(z)} = \frac{1 + \phi(z)}{1 - q\phi(z)}. \tag{2.9}$$

The equality (2.9) shows that

$$\frac{tz^2 D_q f(z)}{g(z)\, g(tz)} \prec \frac{1+z}{1-qz}, \ z \in U.$$

Conversely, let $\frac{tz^2 D_q f(z)}{g(z)g(tz)} \prec \frac{1+z}{1-qz}$. We have

$$\frac{tz^2 D_q f(z)}{g(z)\, g(tz)} = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{M}\right)\phi(z)}, \ M = \frac{1}{1-q}, \ M > 1, |t| \leq 1, t \neq 0, z \in U.$$

So,

$$\frac{tz^2 D_q f(z)}{g(z)\,g(tz)} - M = M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}.$$

On the other hand, the function $\left(\frac{\frac{1-M}{M}+\phi(z)}{1+\frac{1-M}{M}\phi(z)}\right)$, with $\phi(0) = 0$ and $|\phi(z)| < 1$ maps the unit circle onto itself, so that

$$\begin{aligned}\left|\frac{tz^2 D_q f(z)}{g(z)\,g(tz)} - M\right| &= \left|M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}\right| < M \Rightarrow \\ \left|\frac{tz^2 D_q f(z)}{g(z)\,g(tz)} - \frac{1}{1-q}\right| &\le \frac{1}{1-q},\ q \in (0,1), |t| \le 1, t \ne 0, z \in U,\end{aligned}$$

and the proof is now complete. □

Remark 2.2.2. *If we use the notation (2.3), the relation (2.6) is equivalent to*

$$\frac{zD_q f(z)}{G(z)} \prec \frac{1+z}{1-qz},\ q \in (0,1),\ z \in U. \tag{2.10}$$

Corollary 2.2.3. *Let $f \in K_{t,q}$ with respect to the function $g \in S^*\left(\frac{1}{2}\right)$. Then*

$$\frac{1-r}{1+qr} \le \left|\frac{tz^2 D_q f(z)}{g(z)\,g(tz)}\right| \le \frac{1+r}{1-qr},\ q \in (0,1), |t| \le 1, t \ne 0, z \in U. \tag{2.11}$$

Proof. The linear transformation $\omega(z) = \frac{1+z}{1-qz}$ maps $|z| = r$ onto the circle with the centre $C\left(\frac{1+qr^2}{1-q^2r^2}, 0\right)$ and the radius $\rho(r) = \frac{(1+q)r}{1-q^2r^2}$. Using the subordination principle, we obtain

$$\left|\frac{tz^2 D_q f(z)}{g(z)\,g(tz)} - \frac{1+qr^2}{1-q^2r^2}\right| \le \frac{(1+q)\,r}{1-q^2r^2},\ q \in (0,1), |t| \le 1, t \ne 0, z \in U,$$

which implies the desired result. □

The next step is to obtain coefficient inequalities for functions in the class $K_{t,q}$. For this purpose we apply a technique due to Clunie, which is widely used in geometric function theory to handle expressions involving Schwarz functions.

Theorem 2.2.4. *Let $f \in K_{t,q}$ with respect to the function g, f given by (2.5), $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $g \in S^*\left(\frac{1}{2}\right)$. Then*

$$\sum_{n=1}^{k}\left|a_n \frac{1-q^n}{1-q} - c_n\right|^2 \le \sum_{n=1}^{k-1}\left|a_n \frac{q-q^{n+1}}{1-q} + c_n\right|^2, \tag{2.12}$$

where $a_1 = 1, c_n = \sum_{j=1}^{n} b_j b_{n-j+1} t^{n-j}$, with $b_1 = 1$, $|t| \le 1$, $t \ne 0$, $q \in (0,1)$, $z \in U$.

Proof. Using Theorem 2.2.1 and Remark 2.2.2, we have

$$\begin{aligned}\frac{zD_q f(z)}{G(z)} &= \frac{1+\phi(z)}{1-q\phi(z)} \Leftrightarrow z\,(D_q f)(z) - zq\phi(z)\,D_q f(z) = G(z) + G(z)\,\phi(z) \Leftrightarrow \\ zD_q f(z) - G(z) &= \phi(z)\,(G(z) + zqD_q f(z)).\end{aligned}$$

From the definition of $D_q f(z)$, we get

$$\sum_{n=1}^{\infty}\left(a_n\frac{1-q^n}{1-q}-c_n\right)z^j=\phi(z)\sum_{n=1}^{\infty}\left(a_n\frac{q-q^{n+1}}{1-q}+c_n\right)z^n,$$

where $a_1=1, c_n=\sum_{j=1}^{n}b_jb_{n-j+1}t^{n-j}$, with $b_1=1$, $|t|\le 1$, $t\neq 0$, $q\in(0,1)$, $z\in U$. Thus,

$$\sum_{n=1}^{k}\left(a_n\frac{1-q^n}{1-q}-c_n\right)z^n+\sum_{n=k+1}^{\infty}d_nz^n=\phi(z)\sum_{n=1}^{k-1}\left(a_n\frac{q-q^{n+1}}{1-q}+c_n\right)z^n,$$

where the sum $\sum_{n=k+1}^{\infty}d_nz^n$ is convergent in U. Let $z=re^{i\theta}$. Since $|\phi(z)|<1$, we deduce that

$$\sum_{n=1}^{k}\left|a_n\frac{1-q^n}{1-q}-c_n\right|^2r^{2k}\le\sum_{n=1}^{k-1}\left|a_n\frac{q-q^{n+1}}{1-q}+c_n\right|^2r^{2k}\ . \tag{2.13}$$

Passing to the limit in (2.13) as $r\to 1$, we obtain the inequality (2.12), which completes our proof. ☐

This proof is based on a method introduced by Clunie (see [9]).

As a direct consequence of the above argument, we obtain the following inequality when the auxiliary function g is chosen to be the identity.

Corollary 2.2.5. *Let $f\in K_{t,q}$ with respect to the function g, f given by (2.5) and $g(z)=z$, $g\in S^*\left(\frac{1}{2}\right)$. Then*

$$\sum_{n=2}^{k}|a_n|^2\left(\frac{1-q^n}{1-q}\right)^2\le(1+q)^2+\sum_{n=2}^{k-1}|a_n|^2\left(\frac{q-q^{n+1}}{1-q}\right)^2, \tag{2.14}$$

where $|t|\le 1$, $t\neq 0$, $q\in(0,1)$, $z\in U$.

Next, we give a sufficient condition for functions to belong to the class $K_{t,q}$.

Theorem 2.2.6. *Let $f\in\mathcal{A}$, given by (2.5), $g(z)=z+\sum_{n=2}^{\infty}b_nz^n, g\in S^*\left(\frac{1}{2}\right)$, and $c_n=\sum_{j=1}^{n}b_jb_{n-j+1}t^{j-1}$, with $b_1=1$, $|t|\le 1$, $t\neq 0$. If*

$$\sum_{n=2}^{\infty}\left(\left|\frac{c_n}{1-q}-a_n[n]_q\right|+\frac{|c_n|}{1-q}\right)\le 1,\ \ q\in(0,1),z\in U, \tag{2.15}$$

then $f\in K_{t,q}$ with respect to g.

Proof. If $f\in K_{t,q}$, the relation (2.1) is equivalent to

$$\frac{q+\sum_{n=2}^{\infty}|c_n-a_n(1-q^n)|}{1-\sum_{n=2}^{\infty}|c_n|}\le 1,\ \ q\in(0,1),|t|\le 1,t\neq 0,z\in U. \tag{2.16}$$

From (2.16) we obtain

$$\frac{\left|qz+\sum\limits_{n=2}^{\infty}\left(c_n+a_nq^n-a_n\right)z^n\right|}{\left|z+\sum\limits_{n=2}^{\infty}c_nz^n\right|}\leq 1,\ q\in(0,1),|t|\leq 1,t\neq 0,z\in U,$$

or equivalently,

$$\left|\frac{zD_qf(z)}{G(z)}-\frac{1}{1-q}\right|\leq\frac{1}{1-q},\ q\in(0,1),|t|\leq 1,t\neq 0,z\in U. \tag{2.17}$$

Therefore, if the function f satisfies the inequality (2.15), then $f\in K_{t,q}$ with respect to g.
The proof of our theorem is now complete. □

Remark 2.2.7. *In the particular case when $f_0(z)=z+\frac{z^2}{1+q}$ and $g_0(z)=z$, $|t|\leq 1$, $t\neq 0$, $q\in(0,1)$, $z\in U$, f_0 belongs to the class $K_{t,q}$ with respect to g_0 (see Example 2.1.2). But*

$$\sum_{n=2}^{\infty}\left(\left|\frac{c_n}{1-q}-a_n[n]_q\right|+\frac{|c_n|}{1-q}\right)=\frac{1}{1-q}[2]_q=1+q\nleq 1,\quad q\in(0,1).$$

This shows that (2.15) is only a sufficient condition.

2.3 Bieberbach - de Branges Theorem for the class $K_{t,q}$

We now turn to coefficient growth estimates of Bieberbach–de Branges type. These estimates describe the sharp upper bounds for $|a_n|$ in the class $K_{t,q}$ and generalize classical results for close-to-convex functions.

We need the following result:

Lemma 2.3.1. *A function $f\in K_{t,q}$ with respect to g if and only if there exists a function $g\in S^*\left(\frac{1}{2}\right)$ such that*

$$\frac{|g(z)g(tz)+tz(f(qz)-f(z))|}{|g(z)g(tz)|}\leq 1,\ |t|\leq 1,t\neq 0,q\in(0,1),z\in U. \tag{2.18}$$

Proof. The proof can be obtained from $\frac{tz^2D_qf(z)}{g(z)g(tz)}=\frac{tz(f(qz)-f(z))}{g(z)g(tz)(q-1)}$. □

If we use the notation (2.3), the inequality (2.18) is equivalent to

$$\frac{|G(z)+f(qz)-f(z)|}{|G(z)|}\leq 1,\ q\in(0,1),z\in U. \tag{2.19}$$

We now proceed to state and prove the Bieberbach - de Branges Theorem for the generalized class $K_{t,q}$. The Bieberbach - de Branges conjecture for close-to convex functions is proved by Reade (see [11]). It states that if $f\in K$, then $|a_n|\leq n$, for all $n\geq 2$. The Bieberbach - de Branges Theorem for the class of q-close-to-convex functions is proved in ([12]).

Theorem 2.3.2. *Let $f\in K_{t,q}$ with respect to g, f given by (2.5), $g(z)=z+\sum\limits_{n=2}^{\infty}b_nz^n$, $g\in S^*\left(\frac{1}{2}\right)$, then*

$$|a_n|\leq\frac{1-q}{1-q^n}\left[n+(1+q)\frac{n(n-1)}{2}\right],\quad \textit{for all } n\geq 2. \tag{2.20}$$

Proof. Since $f \in K_{t,q}$, by (2.19), there exists a function $w : U \to \overline{U}$, $w(z) = q + \sum_{n=2}^{\infty} w_n z^n$ such that

$$G(z) + f(qz) - f(z) = w(z) G(z). \tag{2.21}$$

Evidently, $w(0) = q$. By assuming $a_1 = c_1 = 1$, we have

$$\sum_{n=1}^{\infty} (c_n + a_n q^n - a_n) z^j = \sum_{n=1}^{\infty} q c_n z^n + \sum_{n=2}^{\infty} \left(\sum_{n=1}^{n-1} w_{n-1} c_n \right) z^n, \ q \in (0,1).$$

Equating the coefficients of z^j, for $n \geq 2$, we obtain

$$a_n (q^n - 1) = c_n (q-1) + \sum_{n=1}^{n-1} w_{n-1} c_n, \ q \in (0,1).$$

It is easy to verify that $|w_n| \leq 1 - |w_0^2| = 1 - q^2$, for all $n \geq 1$, $q \in (0,1)$. Since $G(z) \in S^*$, then $|c_n| \leq n$, for all $n \geq 2$. Therefore we get the bound

$$|a_n| \leq \frac{1-q}{1-q^n} \left[n + (1+q) \sum_{n=1}^{n-1} n \right], \ q \in (0,1), \text{ for all } n \geq 2,$$

which shows that the desired assertion (2.20) holds. □

Corollary 2.3.3. *Let $f \in K_{t,q}$ with respect to $g = \frac{z}{1-z} \in S^* \left(\frac{1}{2}\right)$, f given by (2.5), $q \in (0,1)$, $|t| \leq 1$, $t \neq 0$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1}{[n]_q} \frac{1}{1-t} \left[(1+q)(n+1) + \frac{(1-t^n)(t+q)}{t-1} \right], \ q \in (0,1). \tag{2.22}$$

Proof. By rewritting the function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$, we get

$$G(z) = \frac{g(z) g(tz)}{tz} = \sum_{n=1}^{\infty} \frac{t^n - 1}{t-1} z^n, \quad |t| \leq 1, t \neq 0, z \in U.$$

From (2.5), we obtain

$$\sum_{n=1}^{\infty} \frac{t^n - 1}{t-1} z^n + (q-1) z + \sum_{n=2}^{\infty} (q^n - 1) a_n z^n = \left(q + \sum_{n=2}^{\infty} w_n z^n \right) \sum_{n=1}^{\infty} . \frac{t^n - 1}{t-1} z^n.$$

This is equivalent to

$$\sum_{n=2}^{\infty} (q^n - 1) a_n z^n = (1-q) z + (q-1) \sum_{n=1}^{\infty} . \frac{t^n - 1}{t-1} z^n + \sum_{n=1}^{\infty} . \frac{t^n - 1}{t-1} z^n \sum_{n=2}^{\infty} w_n z^n$$

Equating the coefficients of z^n on both sides of (2.5) for $n \geq 2$, we have

$$(q^n - 1) a_n = (q-1) . \frac{t^n - 1}{t-1} + w_2 \frac{t^n - 1}{t-1} + w_3 \frac{t^{n-1} - 1}{t-1} + ... + w_{n-1}.$$

Since $|w_n| \leq 1 - |w_0^2| = 1 - q^2$, for all $n \geq 1$, $q \in (0,1)$, $|t| \leq 1$, $t \neq 0$ we get

$$(q^n - 1) |a_n| \leq (q-1) . \frac{t^n - 1}{t-1} + \left(q^2 - 1\right) \left(\frac{t^n - 1}{t-1} + \frac{t^{n-1} - 1}{t-1} + ... + 1 \right),$$

which implies

$$|a_n| \leq \frac{1-q}{1-q^n}\left[\frac{1-t^n}{1-t}+(1+q)\sum_{k=1}^{n-1}\frac{1-t^{n-1}}{1-t}\right], \text{ for } n \geq 2.$$

Or, equivalently,

$$|a_n| \leq \frac{1}{[n]_q}\frac{1}{1-t}\left[(1+q)(n+1)+\frac{(1-t^n)(t+q)}{t-1}\right], \text{ for } n \geq 2.$$

This completes the proof. □

If we consider $g(z) = z$ in Theorem 2.3.2, we obtain:

Corollary 2.3.4. *Let $f \in K_{t,q}$ with respect to $g(z) = z \in S^*\left(\frac{1}{2}\right)$, f given by (2.5), $q \in (0,1)$, $|t| \leq 1$, $t \neq 0$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1+q}{[n]_q}, \ q \in (0,1), \textit{ for all } n \geq 2. \tag{2.23}$$

In what follows, we turn from classes defined directly by q-difference inequalities, such as $K_{t,q}$, to classes generated by q-differential operators built from special functions. In particular, we focus on operators associated with normalized Jackson q -Bessel functions of the second and third kind and investigate their role in characterizing q-starlike functions of order α. This provides a complementary viewpoint on how q-calculus interacts with geometric function theory.

2.4 q-Bessel Functions and Associated Differential Operators

The Jackson's second and Hahn - Exton (or third Jackson) q-Bessel functions are defined by (see [6])

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{j\geq 0}\frac{(-1)^j\left(\frac{z}{2}\right)^{2j+\nu}}{(q;q)_j\,(q^{\nu+1};q)_j}q^{j(j+\nu)} \tag{2.24}$$

and

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{j\geq 0}\frac{(-1)^j(z)^{2j+\nu}}{(q;q)_j\,(q^{\nu+1};q)_j}q^{\frac{1}{2}j(j+\nu)}, \tag{2.25}$$

where $z \in \mathbb{C}$, $\nu > -1$, $0 < q < 1$ and

$$(a;q)_0 = 1, (a;q)_j = \prod_{k=1}^{j}\left(1-aq^{k-1}\right), (a;q)_\infty = \prod_{k\geq 1}\left(1-aq^{k-1}\right).$$

These analytic functions are q-extensions of the classical Bessel functions of the first kind J_ν. Properties of the above q-extensions of Bessel functions can be found in [3] and [7] and in the references therein. Because neither $J_\nu^{(2)}(z;q)$, nor $J_\nu^{(3)}(z;q)$ belongs to the class $\mathcal{A}$, we consider the following normalized forms (see [2]):

$$h_\nu^{(2)}(z;q) = 2^\nu c_\nu(q)\, z^{1-\frac{\nu}{2}} J_\nu^{(2)}\left(\sqrt{z};q\right) = \sum_{j\geqslant 0} K_j z^{j+1}, \tag{2.26}$$

and

$$h_{\nu}^{(3)}(z;q)=c_{\nu}(q)\,z^{1-\frac{\nu}{2}}J_{\nu}^{(3)}\left(\sqrt{z};q\right)=\sum_{j\geq 0}T_j z^{j+1},\tag{2.27}$$

where $c_{\nu}(q)=\frac{(q;q)_{\infty}}{(q^{\nu+1};q)_{\infty}}$, $K_j=\dfrac{(-1)^j q^{j(j+\nu)}}{4^j (q;q)_j (q^{\nu+1};q)_j}$, $T_j=\dfrac{(-1)^j q^{\frac{1}{2}j(j+1)}}{(q;q)_j (q^{\nu+1};q)_j}$ and $\nu>-1$, $0<q<1$, $z\in\mathbb{C}$.

Clearly, the above functions $h_{\nu}^{s}(z;q)$, $s\in\{2,3\}$, belong to the class $\mathcal{A}$.

We now define the following two differential operators:

$$\begin{aligned}
H_{\nu,\lambda}^{(2),0}(q)f(z) &= f(z)*h_{\nu}^{(2)}(z;q),\\
H_{\nu,\lambda}^{(2),1}(q)f(z) &= (1-\lambda)f(z)*h_{\nu}^{(2)}(z;q)+\lambda z D_q\left(f(z)*h_{\nu}^{(2)}(z;q)\right),\\
&\;\;\;\cdots\cdots\\
H_{\nu,\lambda}^{(2),m}(q)f(z) &= H_{\nu,\lambda}^{(2),1}\left(H_{\nu,\lambda}^{(2),m-1}(q)f(z)\right)=\\
&= z+\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m K_{k-1}a_k z^k,
\end{aligned}\tag{2.28}$$

for $\lambda\geqslant 0,\nu>-1$, $0<q<1$, $z\in\mathbb{C}$, $m\in\mathbb{N}$, where $*$ denotes the usual Hadamard product of analytic functions,

$$K_{k-1}=\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}$$

and

$$\begin{aligned}
H_{\nu,\lambda}^{(3),0}(q)f(z) &= f(z)*h_{\nu}^{(3)}(z;q),\\
H_{\nu,\lambda}^{(3),1}(q)f(z) &= (1-\lambda)f(z)*h_{\nu}^{(3)}(z;q)+\lambda z D_q\left(f(z)*h_{\nu}^{(3)}(z;q)\right),\\
&\;\;\;\cdots\cdots\\
H_{\nu,\lambda}^{(3),m}(q)f(z) &= H_{\nu,\lambda}^{(3),1}\left(H_{\nu,\lambda}^{(3),m-1}(q)f(z)\right)=\\
&= z+\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m T_{k-1}a_k z^k,
\end{aligned}\tag{2.29}$$

for $\lambda\geqslant 0,\nu>-1$, $0<q<1$, $z\in\mathbb{C}$, $m\in\mathbb{N}$, where $*$ denotes the usual Hadamard product of analytic functions and

$$T_{k-1}=\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}.$$

2.5 Geometric Properties of q-Starlike Classes Defined via q-Bessel Operators

The next theorem provides a necessary and sufficient condition for the transform $H_{\nu,\lambda}^{(2),m}(q)f(z)$ to be q-starlike of order α. Such characterizations generalize classical results for Bessel-type operators.

Theorem 2.5.1. *The function $H_{\nu,\lambda}^{(2),m}(q)f(z)\in S_q^*(\alpha)$ if and only if*

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)}\prec\frac{1+z\left[1-\alpha(1+q)\right]}{1-qz},\ z\in U,\ 0\leq\alpha<1,\ 0<q<1,\lambda\geqslant 0,\nu>-1.$$

Proof. Assuming that $h_{\nu}^{2}(z;q) \in S_q^*(\alpha)$, we have:

$$\left|\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} - \frac{1-\alpha q}{1-q}\right| \leq \frac{1-\alpha}{1-q} \Leftrightarrow \left|\frac{1-q}{1-\alpha}\cdot\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} - \frac{1-\alpha q}{1-q}\right| \leq 1.$$

Therefore, the function

$$\varphi(z) = \frac{1-q}{1-\alpha}\cdot\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} - \frac{1-\alpha q}{1-q}$$

has modulus at most 1 in the unit disk U and $\varphi(0) = -q$.

Let

$$\Phi(z) = \frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)} = \frac{\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)} - 1}{[1-\alpha(1+q)] + q\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)}}.$$

The function $\Phi(z)$ satisfies the conditions of Schwarz lemma, i.e. $\Phi(0) = 0,\ |\Phi(0)| < 1$, so, we have:

$$|\Phi(z)| \leq z.$$

We obtain

$$\Phi(z)[1-\alpha(1+q)] + \Phi(z)\,q\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} = \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} - 1.$$

So,

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} = \frac{1+\Phi(z)[1-\alpha(1+q)]}{1-q\Phi(z)}.$$

The above equality shows that

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} \prec \frac{1+z[1-\alpha(1+q)]}{1-qz}.$$

Conversely, let

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} \prec \frac{1+z[1-\alpha(1+q)]}{1-qz}.$$

Then, we have

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} \prec \frac{1+\Phi(z)[1-\alpha(1+q)]}{1-q\Phi(z)} \Rightarrow$$

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)\,f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)\,f(z)} - \frac{1-\alpha q}{1-q} = \frac{1+\Phi(z)[1-\alpha(1+q)]}{1-q\Phi(z)} - \frac{1-\alpha q}{1-q} =$$

$$= \frac{\Phi(z)(1-\alpha) - q(1-\alpha)}{(1-q\Phi(z))(1-q)} = \frac{1-\alpha}{1-q} \cdot \frac{-q+\Phi(z)}{1-q\Phi(z)}.$$

The function $\frac{-q+\Phi(z)}{1-q\Phi(z)}$, with $\Phi(z) = 0$, $|\Phi(z)| \leq 1$, maps the unit disk into itself, so

$$\left| \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q) f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} - \frac{1-\alpha q}{1-q} \right| = \left| \frac{1-\alpha}{1-q} \cdot \frac{-q+\Phi(z)}{1-q\Phi(z)} \right| < \frac{1-\alpha}{1-q}$$

and the proof is now complete. □

Theorem 2.5.2. *The function* $H_{\nu,\lambda}^{(3),m}(q) f(z) \in S_q^*(\alpha)$ *if and only if*

$$\frac{zD_q\left(H_{\nu,\lambda}^{(3),m}(q) f(z)\right)}{H_{\nu,\lambda}^{(3),m}(q) f(z)} \prec \frac{1+z[1-\alpha(1+q)]}{1-qz}, z \in U,\ 0 \leq \alpha < 1,\ 0 < q < 1, \lambda \geqslant 0, \nu > -1.$$

Proof. The proof is similar to the proof of Theorem 2.5.1. □

Corollary 2.5.3. *Let* $H_{\nu,\lambda}^{(2),m}(q) f(z) \in S_q^*(\alpha)$. *Then*

$$\frac{1-r[1-\alpha(1+q)]}{1+qr} \leq \left| \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q) f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} \right| \leq \frac{1+r[1-\alpha(1+q)]}{1-qr},$$

for $z \in U,\ 0 \leq \alpha < 1,\ 0 < q < 1, \lambda \geqslant 0, \nu > -1.$

Proof. The linear transformation

$$\omega(z) = \frac{1+z[1-\alpha(1+q)]}{1-qz}$$

maps $|z| = r$ onto the circle with the center $C(r) = (c, 0)$, where

$$c = \frac{\omega(r) + \omega(-r)}{2} = \frac{1}{2}\left[\frac{1+r[1-\alpha(1+q)]}{1-qr} + \frac{1-r[1-\alpha(1+q)]}{1+qr}\right] =$$

$$= \frac{1+r^2 q[1-\alpha(1+q)]}{1-q^2r^2}.$$

and the radius

$$\rho(r) = \frac{1+r[1-\alpha(1+q)]}{1-qr} - \frac{1+r^2q - \alpha r^2 q(1+q)}{1-q^2r^2} =$$

$$= \frac{r(1-\alpha)(1+q)}{1-q^2r^2}.$$

Using the subordination principle, we get

$$\left| \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q) f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} - \frac{1+qr^2[1-\alpha(1+q)]}{1-q^2r^2} \right| \leq \frac{r(1-\alpha)(1+q)}{1-q^2r^2},$$

which readily yields

$$\frac{1-r[1-\alpha(1+q)]}{1+qr} \leq \left| \frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q) f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} \right| \leq \frac{1+r[1-\alpha(1+q)]}{1-qr}.$$

This proves the conclusion of the corollary. □

Corollary 2.5.4. *Let $H_{\nu,\lambda}^{(3),m}(q)f(z)\in S_q^*(\alpha)$. Then*

$$\frac{1-r\left[1-\alpha(1+q)\right]}{1+qr}\le\left|\frac{zD_q\left(H_{\nu,\lambda}^{(3),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(3),m}(q)f(z)}\right|\le\frac{1+r\left[1-\alpha(1+q)\right]}{1-qr}.$$

Proof. The proof is similar to the proof of Corollary 2.5.3. □

Using again Clunie's method, we now derive coefficient inequalities for functions whose transforms under the operator $H_{\nu,\lambda}^{(2),m}(q)$ are q-starlike. These quadratic estimates play an essential role in extending Bieberbach–de Branges type results to the operator setting.

Theorem 2.5.5. *Let $H_{\nu,\lambda}^{(2),m}(q)f(z)\in S_q^*(\alpha)$, then*

$$\sum_{k=2}^{j}\left|\frac{q-q^k}{1-q}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|^2\le$$

$$\le\sum_{k=2}^{j-1}\left|\left(\frac{1-q^{k+1}}{1-q}-\alpha(1+q)\right)\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|^2,$$

$j=2,3,\ldots$

Proof. By using de definition of $D_qf(z)$ we get

$$zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)=z+\sum_{k=2}^{\infty}\frac{1-q^k}{1-q}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k. \tag{2.30}$$

On the other hand,

$$\frac{zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)}{H_{\nu,\lambda}^{(2),m}(q)f(z)}=\frac{1+\Phi(z)\left[1-\alpha(1+q)\right]}{1-q\Phi(z)}.$$

It follows that

$$zD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)-H_{\nu,\lambda}^{(2),m}(q)f(z)= \tag{2.31}$$

$$=\Phi(z)\left[H_{\nu,\lambda}^{(2),m}(q)f(z)\left(1-\alpha(1+q)\right)+qzD_q\left(H_{\nu,\lambda}^{(2),m}(q)f(z)\right)\right] \tag{2.32}$$

Using (2.28) and (2.29), we obtain

$$z+\sum_{k=2}^{\infty}\frac{1-q^k}{1-q}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k-$$

$$-z-\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k$$

$$=\Phi(z)\left[\left(1-\alpha(1+q)\right)\left(z+\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k\right)+\right.$$

$$\left.+qz+\sum_{k=2}^{\infty}\frac{1-q^k}{1-q}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}qa_kz^k\right]+$$

$$
\begin{aligned}
&+\sum_{k=2}^{\infty}\left(\frac{1-q^{k}}{1-q}-1\right)\left[1+\left([k]_{q}-1\right)\lambda\right]^{m}\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_{k}z^{k}=\\
=\;&\Phi(z)\left[(1+q)(1-\alpha)z+\right.\\
&\left.+\sum_{k=2}^{\infty}\left(\frac{q-q^{k+1}}{1-q}+1-\alpha(1+q)\right)\left[1+\left([k]_{q}-1\right)\lambda\right]^{m}\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_{k}z^{k}\right].
\end{aligned}
$$

So,

$$
\begin{aligned}
&\sum_{k=2}^{\infty}\frac{q-q^{k}}{1-q}\left[1+\left([k]_{q}-1\right)\lambda\right]^{m}\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_{k}z^{k}=\\
=\;&\Phi(z)\left[(1+q)(1-\alpha)z+\right.\\
&\left.+\sum_{k=2}^{\infty}\left(\frac{1-q^{k+1}}{1-q}-\alpha(1+q)\right)\left[1+\left([k]_{q}-1\right)\lambda\right]^{m}\frac{(-1)^{k-1}q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_{k}z^{k}\right].
\end{aligned}
$$

By Clunie's method (see [8]) we obtain the required inequality. □

Theorem 2.5.6. *Let $H_{\nu,\lambda}^{(3),m}(q)\,f(z)\in S_q^{*}(\alpha)$, then*

$$
\begin{aligned}
&\sum_{k=2}^{j}\left|\frac{q-q^{k}}{1-q}\left[1+\left([k]_{q}-1\right)\lambda\right]^{m}\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_{k}\right|^{2}\leq\\
\leq\;&\sum_{k=2}^{j-1}\left|\left(\frac{1-q^{k+1}}{1-q}-\alpha(1+q)\right)\left[1+\left([k]_{q}-1\right)\lambda\right]^{m}\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_{k}\right|^{2},\; j=2,3,...
\end{aligned}
$$

Proof. The proof is similar to the proof of Theorem 2.5.5. □

The following results establish Bieberbach–de Branges type inequalities for the classes generated by $H_{\nu,\lambda}^{(2),m}(q)$ and $H_{\nu,\lambda}^{(3),m}(q)$. They provide explicit upper bounds in terms of the operator parameters and the q-starlikeness order.

The Bieberbach conjecture for the class S_q^{*} is proved in [12].

A necessary and sufficient condition for functions $f(z)$ to be in $S_q^{*}(\alpha)$ was obtained in [1]:

Theorem 2.5.7. *A function $f(z)\in S_q^{*}(\alpha)$ if and only if*

$$
\left|\frac{f(qz)}{f(z)}-\alpha q\right|\leq 1-\alpha,\; z\in U.
$$

Proof. From $\left|\dfrac{zD_qf(z)}{f(z)}\right|=\dfrac{1}{1-q}\left(1-\dfrac{f(qz)}{f(z)}\right)$ and from the definition of $S_q^{*}(\alpha)$ we get the result. □

By using this result, we will analyse the Bieberbach - de Branges theorem for the class of q-starlike functions of order α.

Theorem 2.5.8. *If $H_{\nu,\lambda}^{(2),m}(q)\,f(z)\in S_q^{*}(\alpha)$, then for all $k\geq 2$, we have*

$$
|a_k|\leq\frac{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}{\left[1+\left([k]_{q}-1\right)\lambda\right]^{m}q^{(k-1)(k+\nu-1)}}\cdot\frac{(1-q^{2})(1-\alpha)}{q-q^{k}}\cdot\prod_{l=2}^{k-1}\left(1+\frac{(1-q^{2})(1-\alpha)}{q-q^{l}}\right),
$$

for $z\in U,\; 0\leq\alpha<1,\; 0<q<1,\lambda\geqslant 0,\nu>-1.$

Proof. Let

$H_{\nu,\lambda}^{(2),m}(q) f(z) \in S_q^*(\alpha) \Leftrightarrow \left| \frac{H_{\nu,\lambda}^{(2),m}(q) f(qz)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} - \alpha q \right| \leq 1 - \alpha \Leftrightarrow$

$\left| \frac{\frac{H_{\nu,\lambda}^{(2),m}(q) f(qz)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} - \alpha q}{1-\alpha} \right| \leq 1.$

Then there exists $\varpi : U \to \overline{U}$ such that $\varpi(z) = \frac{\frac{H_{\nu,\lambda}^{(2),m}(q) f(qz)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} - \alpha q}{1-\alpha}$.

Clearly, $\varpi(0) = q$.

It follows that

$$\varpi(z)(1-\alpha) = \frac{H_{\nu,\lambda}^{(2),m}(q) f(qz)}{H_{\nu,\lambda}^{(2),m}(q) f(z)} - \alpha q,$$

so,

$$H_{\nu,\lambda}^{(2),m}(q) f(qz) = \varpi(z) H_{\nu,\lambda}^{(2),m}(q) f(z)(1-\alpha) + \alpha q H_{\nu,\lambda}^{(2),m}(q) f(z).$$

For $a_1 = 1,\ \varpi_0 = q$ we get

$$\sum_{k=1}^{\infty} \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k q^k z^k =$$

$$= \left(\sum_{k=0}^{\infty} \varpi_k z^k\right)\left(\sum_{k=1}^{\infty} \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k\right) \cdot$$

$$\cdot (1-\alpha) + \alpha q \sum_{k=1}^{\infty} \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k =$$

$$= \left((1-\alpha)\sum_{k=0}^{\infty} \varpi_k z^k + \alpha q\right) \sum_{k=1}^{\infty} \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k z^k.$$

Comparing the coefficients of z^k $(k \geq 2)$, we get

$$\left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k q^k =$$

$$= \alpha \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k q +$$

$$+ (1-\alpha)\left[1 + \left([k]_q k - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k q +$$

$$+ (1-\alpha)\sum_{l=1}^{k-1} \varpi_{k-l} \left[1 + \left([l]_q - 1\right)\lambda\right]^m \frac{(-1)^{l-1} q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}} a_l,$$

thus,

$$\left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k \left(q^k - q\right) =$$

$$= (1-\alpha)\sum_{l=1}^{k-1} \varpi_{k-l} \left[1 + \left([l]_q - 1\right)\lambda\right]^m \frac{(-1)^{l-1} q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}} a_l, \text{ for } k \geq 2.$$

Since $|\varpi_k| \leq 1 - |\varpi_0|^2 = 1 - q^2$, for $k \geq 1$,

$$\left| \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k \right| \leq$$
$$\leq \frac{(1-q^2)(1-\alpha)}{q-q^k} \sum_{l=1}^{k-1} \left[1 + \left([l]_q - 1\right)\lambda\right]^m \frac{(-1)^{l-1} q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}} a_l, \ k \geq 2.$$

Thus, for $k = 2$, $\left| [1+\lambda]^m \frac{(-1)\, q^{1+\nu}}{4^{k-1}(q;q)_1 (q^{\nu+1};q)_1} a_2 \right| \leq \frac{(1-q^2)(1-\alpha)}{q-q^2}$, and for $k \geq 3$, by applying a similar method to estimate $|a_{k-1}|$, we obtain

$$\left| \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k \right| \leq$$
$$\leq \frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1 + \frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right) \sum_{l=1}^{k-1} \left[1 + \left([l]_q - 1\right)\lambda\right]^m \frac{(-1)^{l-1} q^{(l-1)(l+\nu-1)}}{4^{l-1}(q;q)_{l-1}(q^{\nu+1};q)_{l-1}} a_l.$$

Iteratively, we conclude that, for $k \geq 3$,

$$\left| \left[1 + \left([k]_q - 1\right)\lambda\right]^m \frac{(-1)^{k-1} q^{(k-1)(k+\nu-1)}}{4^{k-1}(q;q)_{k-1}(q^{\nu+1};q)_{k-1}} a_k \right| \leq$$
$$\leq \frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1 + \frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right)\left(1 + \frac{(1-q^2)(1-\alpha)}{q-q^{k-2}}\right) ... \left(1 + \frac{(1-q^2)(1-\alpha)}{q-q^2}\right),$$

and the proof is now completed. □

Theorem 2.5.9. *(The Bieberbach - de Branges theorem for $S_q^*(\alpha)$) If $H_{\nu,\lambda}^{(3),m}(q) f(z) \in S_q^*(\alpha)$, then for all $k \geq 2$, we have*

$$|a_k| \leq \frac{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}{\left[1 + \left([k]_q - 1\right)\lambda\right]^m q^{\frac{1}{2}k(k-1)}} \cdot \frac{(1-q^2)(1-\alpha)}{q-q^k} \cdot \prod_{l=2}^{k-1}\left(1 + \frac{(1-q^2)(1-\alpha)}{q-q^l}\right),$$

for $z \in U$, $0 \leq \alpha < 1$, $0 < q < 1, \lambda \geqslant 0, \nu > -1$.

Proof. Let

$$H_{\nu,\lambda}^{(3),m}(q) f(z) \in S_q^*(\alpha) \Leftrightarrow \left| \frac{H_{\nu,\lambda}^{(3),m}(q) f(qz)}{H_{\nu,\lambda}^{(3),m}(q) f(z)} - \alpha q \right| \leq 1 - \alpha \Leftrightarrow \left| \frac{\frac{H_{\nu,\lambda}^{(3),m}(q) f(qz)}{H_{\nu,\lambda}^{(3),m}(q) f(z)} - \alpha q}{1-\alpha} \right| \leq 1.$$

Then there exists $\varpi : U \to \overline{U}$ such that $\varpi(z) = \frac{\frac{H_{\nu,\lambda}^{(3),m}(q) f(qz)}{H_{\nu,\lambda}^{(3),m}(q) f(z)} - \alpha q}{1-\alpha}$.

Clearly, $\varpi(0) = q$.

It follows that

$$\varpi(z)(1-\alpha)=\frac{H_{\nu,\lambda}^{(3),m}(q)f(qz)}{H_{\nu,\lambda}^{(3),m}(q)f(z)}-\alpha q,$$

so,

$$H_{\nu,\lambda}^{(3),m}(q)f(qz)=\varpi(z)H_{\nu,\lambda}^{(3),m}(q)f(z)(1-\alpha)+\alpha q H_{\nu,\lambda}^{(3),m}(q)f(z).$$

For $a_1=1,\ \varpi_0=q$ we get

$$\begin{aligned}&\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kq^kz^k=\\ =&\left(\sum_{k=0}^{\infty}\varpi_kz^k\right)\left(\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k\right)(1-\alpha)+\\ &+\alpha q\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k=\\ =&\left((1-\alpha)\sum_{k=0}^{\infty}\varpi_kz^k+\alpha q\right)\sum_{k=1}^{\infty}\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kz^k.\end{aligned}$$

Comparing the coefficients of z^k $(k\geq 2)$, we get

$$\begin{aligned}&\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kq^k=\\ =&\ \alpha\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kq+\\ &+(1-\alpha)\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_kq+\\ &+(1-\alpha)\sum_{l=1}^{k-1}\varpi_{k-l}\left[1+\left([l]_q-1\right)\lambda\right]^m\frac{(-1)^{l-1}q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,\end{aligned}$$

thus,

$$\begin{aligned}&\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\left(q^k-q\right)=\\ =&\ (1-\alpha)\sum_{l=1}^{k-1}\varpi_{k-l}\left[1+\left([l]_q-1\right)\lambda\right]^m\frac{(-1)^{l-1}q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,\ \text{for} k\geq 2.\end{aligned}$$

Since $|\varpi_k|\leq 1-|\varpi_0|^2=1-q^2$, for $k\geq 1$,

$$\begin{aligned}&\left|\left[1+\left([k]_q-1\right)\lambda\right]^m\frac{(-1)^{k-1}q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}(q^{\nu+1};q)_{k-1}}a_k\right|\leq\\ \leq&\ \frac{(1-q^2)(1-\alpha)}{q-q^k}\sum_{l=1}^{k-1}\left[1+\left([l]_q-1\right)\lambda\right]^m\frac{(-1)^{l-1}q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}(q^{\nu+1};q)_{l-1}}a_l,\ k\geq 2.\end{aligned}$$

Thus, for $k=2$, $\left|[1+\lambda]^m\frac{(-1)q}{(q;q)_1(q^{\nu+1};q)_1}a_2\right|\leq\frac{(1-q^2)(1-\alpha)}{q-q^2}$, and for $k\geq 3$, by applying a similar method to estimate $|a_{k-1}|$, we obtain

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}\, q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}\,(q^{\nu+1};q)_{k-1}} a_k\right| \leq$$

$$\leq \frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right)\sum_{l=1}^{k-1}\left[1+\left([l]_q-1\right)\lambda\right]^m \frac{(-1)^{l-1}\, q^{\frac{1}{2}l(l-1)}}{(q;q)_{l-1}\,(q^{\nu+1};q)_{l-1}} a_l.$$

Iteratively, we conclude that, for $k \geq 3$,

$$\left|\left[1+\left([k]_q-1\right)\lambda\right]^m \frac{(-1)^{k-1}\, q^{\frac{1}{2}k(k-1)}}{(q;q)_{k-1}\,(q^{\nu+1};q)_{k-1}} a_k\right| \leq$$

$$\leq \frac{(1-q^2)(1-\alpha)}{q-q^k}\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-1}}\right)\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{k-2}}\right)\cdots\left(1+\frac{(1-q^2)(1-\alpha)}{q-q^{2}}\right),$$

and the proof is now completed. □

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Chapter 3

Higher Order q-Derivatives and Normalized Function Classes

Contents

In this chapter (see [6, 3]), we develop a unified framework for the study of analytic function classes defined by higher-order q-derivatives, extending classical geometric function theory to the setting of q-calculus and preparing the ground for the Janowski-type multivalent classes introduced later in the chapter.

3.1 Higher-Order q-Derivatives and Associated Normalized Function Classes

Let $f \in \mathcal{A}_p$, given by $f(z) = z^p + \sum_{j=1}^{\infty} a_{j+p} z^{j+p}$. We consider the q-differential operator $V_{q,p}^{\xi} : \mathcal{A}_p \to \mathcal{A}_p$, defined by:

$$V_{q,p}^{\xi} f(z) = z^p + \sum_{j=1}^{\infty} [j+1]_q^{\xi} a_{j+p} z^{j+p}. \tag{3.1}$$

This q-differential operator was introduced by convolution in [11].

Remark 3.1.1. *The operator $V_{q,p}^{\xi} f(z)$ corresponds to a generalization that simplifies to the Salagean type q-operator for $p = 1$, originally formulated by Govindaraj and Sivasubramanian [9]. In the limit as $q \to 1^-$ with $p = 1$, the operator specified in (3.1) recovers the well-known Salagean derivative [23]. This shows that the generalized operator naturally extends classical results.*

Differentiating $V_{q,p}^{\xi}$ with respect to the q-derivative, we arrive at:

$$\left(D_q V_{q,p}^{\xi}\right) f(z) = \frac{V_{q,p}^{\xi} f(qz) - V_{q,p}^{\xi} f(z)}{z(q-1)} = [p]_q z^{p-1} + \sum_{j=1}^{\infty} \left([j+1]_q\right)^{\xi} [j+p]_q a_{j+p} z^{j+p-1} \tag{3.2}$$

Remember that, in the first chapter, we introduced the operator:

$$\widetilde{D}_q f(z) = \begin{cases} \frac{f(qz) - f(q^{-1}z)}{z(q-q^{-1})}, z \neq 0, q \neq 1, z \in U; \\ f'(0), \text{ for } z = 0. \end{cases} \tag{3.3}$$

By iterating the q-derivative and using (3.2) together with (3.3), we arrive at:

$$\left(D_q^{(1)} V_{q,p}^{\xi}\right) f(z) = \left(D_q V_{q,p}^{\xi}\right) f(z),$$

$$\left(D_q^{(2)} V_{q,p}^{\xi} f\right)(z) = [p]_q [p-1]_q z^{p-2} + \sum_{j=1}^{\infty} \left([j+1]_q\right)^{\xi} [j+p]_q [j+p-1]_q a_{j+p} z^{j+p-2},$$

$$\left(D_q^{(p)} V_{q,p}^{\xi} f\right)(z) = [p]_q! + \sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q!} \left([j+1]_q\right)^{\xi} a_{j+p} z^{j}. \tag{3.4}$$

Here, $\left(D_q^{(p)} V_{q,p}^{\xi} f\right)(z)$ denotes the p-th order q-derivative of $V_{q,p}^{\xi} f$, where $p \in \mathbb{N}$.

In a more general setting, the k-th q-derivative of $V_{q,p}^{\xi} f$, with $0 \leq k \leq p$, takes the form:

$$\left(D_q^{(\kappa)} V_{q,p}^{\xi} f\right)(z) = \frac{[p]_q!}{[p-k]_q!} z^{p-k} + \sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j+p-k]_q!} \left([j+1]_q\right)^{\xi} a_{j+p} z^{j+p-k}. \tag{3.5}$$

The following basic identity, relating consecutive parameters ξ and $\xi - 1$, will play a crucial role in what follows.

Proposition 3.1.2. *The identity given below is valid for the involved operators and holds for all functions $f \in \mathcal{A}_p$:*

$$z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z) = \left(D_q^{(p-1)} V_{q,p}^{\xi} f\right)(z). \tag{3.6}$$

Proof. Indeed,

$$\begin{aligned} z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z) &= [p]_q!z+\sum_{j=1}^{\infty}\frac{[j+p]_q!}{[j]_q!}\left([j+1]_q\right)^{\xi-1}a_{j+p}z^{j+1} \\ &= [p]_q!z+\sum_{j=1}^{\infty}\frac{[j+p]_q!}{[j+1]_q!}\left([j+1]_q\right)^{\xi}a_{j+p}z^{j+1} \\ &= \left(D_q^{(p-1)}V_{q,p}^{\xi}f\right)(z). \end{aligned}$$

The proof is complete. □

We now use the higher-order q-derivatives of $V_{q,p}^{\xi-1}$ to introduce normalized classes of analytic functions. These classes will be the main objects of study in the first part of the chapter.

Employing the operator $D_q^{(p)}V_{q,p}^{\xi-1}f$, we introduce the ensuing families of holomorphic functions parameterized by $0 \le \beta < 1$, along with a function $\theta \in \mathcal{P}$,

$$\mathcal{S}_{q,p-1}^{\xi-1}(\theta) = \left\{f\in\mathcal{A}:\frac{1}{[p]_q!}D_q^{(p-1)}V_{q,p}^{\xi-1}f\in\mathcal{S}_q(\theta)\right\}, \tag{3.7}$$

$$\mathcal{S}_{q,p-1}^{\xi-1}(\beta,\theta) = \left\{f\in\mathcal{A}:\frac{1}{[p]_q!}D_q^{(p-1)}\mathcal{V}_{q,p}^{\xi-1}f\in\mathcal{S}_q(\beta,\theta)\right\}. \tag{3.8}$$

Specifically, we define

$$\mathcal{S}_{q,p-1}^{\xi-1}\left(\frac{1+Az}{1+Bz}\right)=\mathcal{S}_{q,p-1}^{\xi-1}(A,B),\quad -1<B<A\le 1, \tag{3.9}$$

and

$$\mathcal{S}_{q,p-1}^{\xi-1}\left(\beta,\frac{1+Az}{1+Bz}\right)=\mathcal{S}_{q,p-1}^{\xi-1}(\beta,A,B),\quad -1<B<A\le 1. \tag{3.10}$$

These definitions generalize well-known classes in geometric function theory and allow us to study the effect of the parameter ξ through inclusion relations based on subordination.

3.2 Inclusion, Subordination, and Structural Results

We begin with a lemma of subordination type that will be used repeatedly to compare classes with parameters ξ and $\xi-1$.

Lemma 3.2.1. *[1] Let $\theta\in\mathcal{P}$ and $Re\{\sigma\theta(z)+\tau\}>0$, $\sigma,\tau\in\mathbb{C}$. If χ is holomorphic in U and satisfies the normalization $\chi(0)=1$, it follows that*

$$\chi(z)+\frac{zD_q\chi(z)}{\sigma\chi(z)+\tau}\prec\theta(z),\quad z\in U,$$

implies $\chi(z)\prec\theta(z)$, $z\in U$.

This lemma will allow us to propagate subordination from a differential expression involving χ back to χ itself, and therefore to relate the classes corresponding to different values of ξ.

We now establish inclusion criteria among the classes $\mathcal{S}_{q,p-1}^{\xi}(\theta)$ and $\mathcal{S}_{q,p-1}^{\xi}(\beta,\theta)$,

This part of the chapter focuses on establishing inclusion criteria among the classes $\mathcal{S}_{q,p-1}^{\xi}(\theta)$ and $\mathcal{S}_{q,p-1}^{\xi-1}(\theta)$ and similarly for their β- versions. The key idea is to use higher-order q-derivatives of the operator $V_{q,p}^{\xi}$ together with Lemma 3.2.1 to show that class membership is preserved when passing from the parameter ξ to $\xi-1$. Special choices of θ, such as $\frac{1+z}{1-qz}$, will then yield concrete families that play a central role in geometric function theory.

Theorem 3.2.2. *Let $f \in \mathcal{A}_p$, and $\theta \in \mathcal{P}$. Then the following inclusion property is satisfied:*

$$\mathcal{S}_{q,p-1}^{\xi}(\theta) \subseteq \mathcal{S}_{q,p-1}^{\xi-1}(\theta).$$

Proof. Suppose that f belongs to the class $\mathcal{S}_{q,p}^{\xi}(\theta)$.
We define

$$\frac{zD_q\left(\frac{1}{[p]_q!}D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)}{\left(\frac{1}{[p]_q!}D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} = \chi(z), \tag{3.11}$$

which is equivalent to

$$\frac{zD_q^{(p)}V_{q,p}^{\xi-1}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} = \chi(z), \tag{3.12}$$

under the conditions that $\chi \in \mathcal{H}(U)$, satisfies $\chi(0) = 1$ and $\theta(z)$ does not vanish. Substituting equation (3.6) into (3.12), we obtain:

$$\frac{\left(D_q^{(p-1)}V_{q,p}^{\xi}f\right)(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} = \chi(z). \tag{3.13}$$

Logarithmic q-differentiation of (3.13) yields

$$\frac{z\left(D_q^{(p)}V_{q,p}^{\xi}f\right)(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi}f\right)(z)} = \chi(z) + \frac{zD_q\chi(z)}{\chi(z)}. \tag{3.14}$$

Given that $f \in \mathcal{S}_{q,p-1}^{\xi}(\theta)$, we infer from (3.14) that

$$\chi(z) + \frac{zD_q\chi(z)}{\chi(z)} \prec \theta(z).$$

Through the use of Lemma 3.2.1, it follows that

$$\frac{zD_q^{(p)}V_{q,p}^{\xi-1}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} \prec \theta(z).$$

Consequently, $f \in \mathcal{S}_{q,p-1}^{\xi-1}(\theta)$, completing the proof. □

Theorem 3.2.2 shows that decreasing ξ by one enlarges the class, in the sense that every function belonging to $\mathcal{S}_{q,p-1}^{\xi}(\theta)$ remains in the corresponding class with parameter $\xi-1$. The following corollaries record this inclusion for specific comparison functions.

Corollary 3.2.3. *For $f \in \mathcal{A}_p$, the inclusion below is satisfied:*

$$\mathcal{S}_{q,p-1}^{\xi}(A,B) \subseteq \mathcal{S}_{q,p-1}^{\xi-1}(A,B),$$

where $-1 \leq B < A \leq 1$

Proof. Let $\theta(z) = \frac{1+Az}{1+Bz}$, where $-1 \leq B < A \leq 1$. By applying Theorem 3.2.2, the desired inclusion follows directly. □

Corollary 3.2.4. *Given $f \in \mathcal{A}_p$, the inclusion below follows:*

$$\mathcal{S}_{q,p-1}^{\xi}(1, -q) \subseteq \mathcal{S}_{q,p-1}^{\xi-1}(1, -q).$$

Proof. Taking $\theta(z) = \frac{1+z}{1-qz}$ and applying Theorem 3.2.2, yields the stated outcome. □

Corollary 3.2.5. *Suppose $f \in \mathcal{A}_p$. Then the following inclusion relation holds:*

$$\mathcal{S}_{q,p-1}^{\xi}(0, -q) \subseteq \mathcal{S}_{q,p-1}^{\xi-1}(0, -q).$$

Proof. Let $\theta(z) = \frac{1}{1-qz}$. By applying Theorem 3.2.2, the conclusion follows immediately. □

We next extend this type of inclusion to the more general classes depending on a parameter β.

Theorem 3.2.6. *Let $f \in \mathcal{A}_p$ and $\theta \in \mathcal{S}$ be such that*

$$Re\{\theta(z)\} < \frac{\beta}{1-\beta}. \tag{3.15}$$

Then the inclusion

$$\mathcal{S}_{q,p-1}^{\xi}(\beta, \theta) \subseteq \mathcal{S}_{q,p-1}^{\xi-1}(\beta, \theta). \tag{3.16}$$

holds.

Proof. Assume $f(z) \in \mathcal{S}_{q,p-1}^{\xi}(\beta, \theta)$ and set

$$\chi(z) = \frac{1}{1-\beta}\left(\frac{zD_q\left(\frac{1}{[p]_q!}D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)}{\frac{1}{[p]_q!}\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} - \beta\right), \tag{3.17}$$

provided χ is holomorphic, normalized so that $\chi(0) = 1$. Expression (3.17) is equivalent to

$$\chi(z) = \frac{1}{1-\beta}\left(\frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} - \beta\right),$$

Using identity (3.6), we find

$$\frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} = \frac{D_q^{(p-1)}V_{q,p}^{\xi}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)}.$$

Substituting this into (3.17) we obtain

$$\frac{D_q^{(p-1)}V_{q,p}^{\xi}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} = (1-\beta)\chi(z) + \beta, \tag{3.18}$$

Now, logarithmically q-differentiating equation (3.18), we derive

$$\frac{1}{1-\beta}\left[\frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} - \beta\right] = \chi(z) + \frac{zD_q\chi(z)}{(1-\beta)\chi(z)+\beta}. \tag{3.19}$$

Since the condition $Re\{\theta(z)\} < \frac{\beta}{1-\beta}$ implies $Re\{(1-\beta)\chi(z)+\beta\} > 0$, we can apply Lemma 3.2.1 to (3.19) thereby obtaining

$$f(z) \in \mathcal{S}^{\xi-1}_{q,p-1}(\beta,\theta),$$

as required. □

As before, Theorem 3.2.6 implies the corresponding inclusions for the Janowsky-type subclasses.

Corollary 3.2.7. *For $f \in \mathcal{A}_p$, the inclusion below is satisfied:*

$$\mathcal{S}^{\xi}_{q,p-1}(\beta,A,B) \subseteq \mathcal{S}^{\xi-1}_{q,p-1}(\beta,A,B),$$

where $-1 \le B < A \le 1$.

Proof. Let $\theta(z) = \frac{1+Az}{1+Bz}$, where $-1 \le B < A \le 1$. Applying Theorem 3.2.6, and considering the notation (3.10), the desired inclusion follows directly. □

Corollary 3.2.8. *Given $f \in \mathcal{A}_p$, the inclusion below follows:*

$$\mathcal{S}^{\xi}_{q,p-1}(\beta,1,-q) \subseteq \mathcal{S}^{\xi-1}_{q,p-1}(\beta,1,-q).$$

Proof. Taking $\theta(z) = \frac{1+z}{1-qz}$ and applying Theorem 3.2.6, along with the notation from (3.10), we obtain the result. □

Corollary 3.2.9. *Suppose $f \in \mathcal{A}_p$. Then the following inclusion relation holds:*

$$\mathcal{S}^{\xi}_{q,p-1}(\beta,0,-q) \subseteq \mathcal{S}^{\xi-1}_{q,p-1}(\beta,0,-q).$$

Proof. Let $\theta(z) = \frac{1}{1-qz}$. By applying Theorem 3.2.6, and taking into account notation 3.10 conclusion follows immediately. □

Observation 3.2.10. *The inclusion theorems established in this section illustrate the central role played by subordination in the analysis of the operator-based classes. In particular, the subordination results obtained above highlight how the generalized higher-order q-differential operator creates structural connections between the newly introduced subclasses and well-studied families of analytic functions. In particular, these relations show that the q-calculus framework preserves many of the hierarchical properties known from the classical theory while also providing genuinely new phenomena for $0 < q < 1$.*

Observation 3.2.11. *We note that the method used here relies essentially on subordination. Although duality methods could also be applied to study related questions, we leave such investigations for future work.*

3.3 Coefficient Estimates and Extremal Problems

We now turn from inclusion properties to coefficient estimates. In this subsection, we focus on two particular subclasses, for which we shall derive sharp Fekete–Szegő type inequalities and sufficient coefficient conditions.

For brevity, we introduce the notation

$$\mathcal{S}^{\xi-1}_{q,p-1}\left(\frac{1+z}{1-qz}\right) = \mathcal{S}^{\xi-1}_{q,p-1}(1,-q) = \mathcal{LS}^{\xi-1}_{q,p-1} \tag{3.20}$$

and

$$\mathcal{S}_{q,p-1}^{\xi-1}\left(\beta, \frac{1+z}{1-qz}\right) = \mathcal{S}_{q,p-1}^{\xi-1}(\beta, 1, -q) = \mathcal{LS}_{q,p-1}^{\xi-1}(\beta) \tag{3.21}$$

In what follows, we investigate sharp coefficient inequalities in $\mathcal{LS}_{q,p-1}^{\xi-1}$ and its generalized form $\mathcal{LS}_{q,p-1}^{\xi-1}(\beta)$. By employing the higher order q-derivative of $V_{q,p}^{\xi-1}$, together with subordination techniques and expansions of associated Carathéodory functions, we derive Fekete–Szegő type estimates for the coefficients a_{p+1} and a_{p+2}. Additionally, we provide adequate analytic constraints under which $f \in \mathcal{A}_p$ is contained in these function subclasses, and we analyze special cases obtained by choosing specific values of the parameter ξ. As a preliminary step, we recall the following lemma concerning Carathéodory functions.

Lemma 3.3.1. *[19] Let the function $w(z)$ given by $w(z) = 1 + w_1 z + w_2 z^2 + \cdots$ belonging to $\mathcal{P}$, the set of mappings in U whose range lies in the $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. Thus, for every complex constant K, the relation below is satisfied:*

$$\left|w_2 - K w_1^2\right| \leq \begin{cases} -4K + 2, & K < 0; \\ 2, & 0 \leq K \leq 1; \\ 4K - 2, & K > 1. \end{cases} \tag{3.22}$$

For K outside the interval $[0, 1]$, the identity in (3.22) is valid exclusively when $w(z) = \frac{1+z}{1-z}$, or a rotated instance of it. For K in the interval $(0, 1)$, the identity in (3.22) holds precisely if $w(z)$ assumes the form $\frac{1+z^2}{1-z^2}$ or a rotated counterpart. The relation stated in (3.22) is valid under the condition $K = 0$, if

$$w(z) = \left(\frac{1+\delta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\delta}{2}\right)\frac{1-z}{1+z}, \quad 0 \leq \delta \leq 1,$$

or a rotated analogue. In the case $K = 1$, the identity in (3.22) is fulfilled whenever $w(z)$ is the inverse (in the compositional sense) of a function for which the relation holds in the $K = 0$ scenario.

Using Lemma 4.1.4.1, we are now in a position to derive Fekete–Szegő type bounds for the classes $\mathcal{LS}_{q,p-1}^{\xi-1}$ and $\mathcal{LS}_{q,p-1}^{\xi-1}(\beta)$. We begin with the non-parameterized case.

Theorem 3.3.2. *Consider $f \in \mathcal{A}_p$ belonging to the class $\mathcal{LS}_{q,p-1}^{\xi-1}$. Then*

$$\left|a_{p+2} - \lambda a_{p+1}^2\right| \leq \begin{cases} \frac{[2]_q}{q^2[1+p]_q[2+p]_q\left([3]_q\right)^{\xi-2}} K(q,p,\xi) & \text{if } \lambda < \gamma_1 \text{ ;} \\ \frac{[2]_q}{q[1+p]_q[2+p]_q\left([3]_q\right)^{\xi-2}} & \text{if } \gamma_1 \leq \lambda \leq \gamma_2 \text{ ;} \\ -\frac{[2]_q}{q^2[1+p]_q[2+p]_q\left([3]_q\right)^{\xi-2}} K(q,p,\xi) & \text{if } \gamma_2 < \lambda, \end{cases}$$

where

$$K(q,p,\xi) = \frac{q[1+p]_q\left([2]_q\right)^{2(\xi-2)} + (q^2+1)[1+p]_q\left([2]_q\right)^{2(\xi-2)} - \lambda[2]_q[2+p]_q\left([3]_q\right)^{\xi-2}}{[1+p]_q\left([2]_q\right)^{2(\xi-2)}},$$

$$\gamma_1 = \frac{(q^2+1)[1+p]_q\left([2]_q\right)^{2(\xi-2)}}{[2]_q[2+p]_q\left([3]_q\right)^{\xi-2}},$$

and

$$\gamma_2 = \frac{[1+p]_q \left([2]_q\right)^{2(\xi-1)}}{[2]_q [2+p]_q \left([3]_q\right)^{\xi-2}}.$$

Each of these results is sharp.

Proof. Assume that $f \in \mathcal{LS}_{q,p-1}^{\xi-1}$. By (3.7) and (3.20), it follows that

$$\frac{zD_q^{(p)}V_{q,p}^{\xi-1}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} \prec \frac{1+z}{1-qz}. \tag{3.23}$$

We proceed to define the function $w \in \mathcal{P}$,

$$w(z) = \frac{1+v(z)}{1-v(z)} = 1 + w_1 z + w_2 z^2 + w_3 z^3 + ... \tag{3.24}$$

From (3.24) we deduce that

$$v(z) = \frac{w(z)-1}{w(z)+1}.$$

Hence, from (3.23), it follows that

$$\frac{zD_q^{(p)}V_{q,p}^{\xi-1}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} = \frac{1+v(z)}{1-qv(z)}, \tag{3.25}$$

where

$$\frac{1+v(z)}{1-qv(z)} = \frac{2w(z)}{(1-q)w(z)+(1+q)}. \tag{3.26}$$

So, from (3.25) and (3.26), we obtain

$$\frac{zD_q^{(p)}V_{q,p}^{\xi-1}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} = \frac{2w(z)}{(1-q)w(z)+(1+q)}.$$

Using (3.25), it follows, upon simplification, that

$$\frac{2w(z)}{(1-q)w(z)+(1+q)} = 1 + \frac{1}{2}(q+1)w_1 z + \frac{1}{4}(q+1)\left[w_1^2(q-1)+2w_2\right]z^2 + ...$$

Proceeding similarly, we obtain that

$$\begin{aligned} &\frac{zD_q^{(p)}V_{q,p}^{\xi-1}f(z)}{\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)} \\ &= 1 + [1+p]_q \left([2]_q\right)^{\xi-2} q a_{p+1} z \\ &+ [1+p]_q \left\{[2+p]_q \left([3]_q\right)^{\xi-2} q a_{p+2} - [1+p]_q \left([2]_q\right)^{2(\xi-2)} q a_{p+1}^2\right\} z^2 + ... \end{aligned}$$

Thus, we obtain

$$a_{p+1} = \frac{(q+1)}{2q[1+p]_q \left([2]_q\right)^{\xi-2}} w_1,$$

and

$$a_{p+2} = \frac{[2]_q \left[w_1^2 (q^2+1) + 2q w_2\right]}{4q^2 [1+p]_q [2+p]_q \left([3]_q\right)^{\xi-2}}.$$

It is clear, therefore, that

$$\left|a_{p+2} - \lambda a_{p+1}^2\right| = \frac{[2]_q}{2q [1+p]_q [2+p]_q \left([3]_q\right)^{\xi-2}} \left|w_2 - k w_1^2\right|, \tag{3.27}$$

where

$$k = \frac{\lambda [2]_q [2+p]_q \left([3]_q\right)^{\xi-2} - (q^2+1) [1+p]_q \left([2]_q\right)^{2(\xi-2)}}{2q [1+p]_q \left([2]_q\right)^{2(\xi-2)}}.$$

The desired result is obtained through the application of the Lemma 3.3.1 in (3.27). □

As a direct consequence of Theorem 3.3.2 with $\xi = 1$, the subsequent outcome emerges.

Corollary 3.3.3. *Assume $f \in \mathcal{A}_p$ belongs to the family $\mathcal{LS}_{q,p-1}^{0}$. Then*

$$\left|a_{p+2} - \lambda a_{p+1}^2\right| \leq \begin{cases} \frac{[2]_q [3]_q}{q^2 [1+p]_q [2+p]_q} K(q,p) & \text{if } \lambda < \gamma_1 ; \\ \frac{[2]_q [3]_q}{q [1+p]_q [2+p]_q} & \text{if } \gamma_1 \leq \lambda \leq \gamma_2 ; \\ -\frac{[2]_q [3]_q}{q^2 [1+p]_q [2+p]_q} K(q,p) & \text{if } \gamma_2 < \lambda, \end{cases}$$

where

$$K(q,p) = q^2 + q + 1 - \frac{\lambda [2+p]_q \left([2]_q\right)^3}{[3]_q [1+p]_q},$$

$$\gamma_1 = \frac{(q^2+1) [1+p]_q [3]_q}{\left([2]_q\right)^3 [2+p]_q}$$

and

$$\gamma_2 = \frac{[1+p]_q [3]_q}{[2]_q [2+p]_q}.$$

All of these results represent best-possible cases..

We next extend these Fekete–Szegő type estimates to the generalized class $\mathcal{LS}_{q,p-1}^{\xi-1}(\beta)$.

Theorem 3.3.4. *Assume that $f \in \mathcal{A}_p$ is contained in the family $\mathcal{LS}_{q,p-1}^{\xi-1}(\beta)$. Then*

$$\left|a_{p+2} - \lambda a_{p+1}^2\right| \leq \begin{cases} \frac{[2]_q (1-\beta)}{q^2 [1+p]_q [2+p]_q \left([3]_q\right)^{\xi-2}} K(q,p,\xi,\beta) & \text{if } \lambda < \gamma_3 ; \\ \frac{[2]_q (1-\beta)}{q [1+p]_q [2+p]_q \left([3]_q\right)^{\xi-2}} & \text{if } \gamma_3 \leq \lambda \leq \gamma_4 ; \\ -\frac{[2]_q (1-\beta)}{q^2 [1+p]_q [2+p]_q \left([3]_q\right)^{\xi-2}} K(q,p,\xi,\beta) & \text{if } \gamma_4 < \lambda, \end{cases}$$

where

$$K(q,p,\xi,\beta) = (1-\beta) [2]_q - q^2 + 2q - \frac{\lambda (1-\beta) [2+p]_q \left([3]_q\right)^{\xi-2}}{[1+p]_q \left([2]_q\right)^{2\xi-5}},$$

$$\gamma_3 = \frac{[1+p]_q \left([2]_q\right)^{2\xi-5} \left\{(1-\beta)[2]_q - q(q-1)\right\}}{(1-\beta)[2+p]_q \left([3]_q\right)^{\xi-2}},$$

and

$$\gamma_4 = \frac{[1+p]_q \left([2]_q\right)^{2\xi-5} \left\{(1-\beta)[2]_q - q^2+3q\right\}}{(1-\beta)[2+p]_q \left([3]_q\right)^{\xi-2}}.$$

Each result is the best possible one.

Proof. Let $f \in \mathcal{LS}_{q,p-1}^{\xi-1}(\beta)$. Using (3.7) and (3.21), we obtain that

$$\frac{1}{1-\beta}\left(\frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z)}{\left(D_q^{(p-1)}\mathcal{V}_{q,p}^{\xi-1}f\right)(z)} - \beta\right) \prec \frac{1+z}{1-qz}. \tag{3.28}$$

Define the function $w \in \mathcal{P}$ as follows.

$$w(z) = \frac{1+v(z)}{1-v(z)} = 1 + w_1 z + w_2 z^2 + w_3 z^3 + ... \tag{3.29}$$

It can be deduced from (3.29) that

$$v(z) = \frac{w(z)-1}{w(z)+1}.$$

Consequently, by (3.28), we obtain

$$\frac{1}{1-\beta}\left(\frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z)}{\left(D_q^{(p-1)}\mathcal{V}_{q,p}^{\xi-1}f\right)(z)} - \beta\right) = \frac{1+v(z)}{1-qv(z)}, \tag{3.30}$$

where

$$\frac{1+v(z)}{1-qv(z)} = \frac{2w(z)}{(1-q)w(z)+(1+q)}. \tag{3.31}$$

It follows from (3.30) and (3.31) that

$$\frac{1}{1-\beta}\left(\frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z)}{\left(D_q^{(p-1)}\mathcal{V}_{q,p}^{\xi-1}f\right)(z)} - \beta\right) = \frac{2w(z)}{(1-q)w(z)+(1+q)}.$$

By using (3.30), it can be deduced, after simplification, that

$$\frac{2w(z)}{(1-q)w(z)+(1+q)} = 1 + \frac{1}{2}(q+1)w_1 z + \frac{1}{4}(q+1)\left[w_1^2(q-1)+2w_2\right]z^2 + ...$$

By proceeding in a similar manner, we obtain

$$\frac{1}{1-\beta}\left(\frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z)}{\left(D_q^{(p-1)}\mathcal{V}_{q,p}^{\xi-1}f\right)(z)} - \beta\right) = \frac{z\left(D_q^{(p)}V_{q,p}^{\xi-1}f\right)(z) - \beta\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)}{(1-\beta)\left(D_q^{(p-1)}V_{q,p}^{\xi-1}f\right)(z)}$$

$$= 1 + \frac{[1+p]_q\left([2]_q\right)^{\xi-2}q}{1-\beta}a_{p+1}z$$

$$+ \frac{[1+p]_q}{1-\beta}\left\{[2+p]_q\left([3]_q\right)^{\xi-2}qa_{p+2} - [1+p]_q\left([2]_q\right)^{2(\xi-2)}qa_{p+1}^2\right\}z^2 + ...$$

Hence, we find

$$a_{p+1} = \frac{(1-\beta)(q+1)}{2q[1+p]_q\left([2]_q\right)^{\xi-2}} w_1,$$

and

$$a_{p+2} = \frac{[2]_q\left\{(1-\beta)\left[(1-\beta)[2]_q + q(q-1)\right]w_1^2 + 2qw_2\right\}}{4q^2[1+p]_q[2+p]_q\left([3]_q\right)^{\xi-2}}.$$

Clearly, we have that

$$\left|a_{p+2} - \lambda a_{p+1}^2\right| = \frac{[2]_q(1-\beta)}{2q[1+p]_q[2+p]_q\left([3]_q\right)^{\xi-2}}\left|w_2 - lw_1^2\right|, \tag{3.32}$$

where

$$l = \frac{\lambda(1-\beta)[2]_q[2+p]_q\left([3]_q\right)^{\xi-2} - [1+p]_q\left([2]_q\right)^{2(\xi-2)}\left\{[2]_q(1-\beta) - q(q-1)\right\}}{2q[1+p]_q\left([2]_q\right)^{2(\xi-2)}}.$$

Applying Lemma 3.3.1 in (3.32) yields the desired result. □

Choosing $\xi = 1$ in Theorem 3.3.4 leads directly to the subsequent result.

Corollary 3.3.5. *Suppose that the function $f \in \mathcal{A}_p$ lies within the subclass $\mathcal{LS}_{q,p-1}^{0}(\beta)$. Then*

$$\left|a_{p+2} - \lambda a_{p+1}^2\right| \leq \begin{cases} \frac{[2]_q[3]_q(1-\beta)}{q^2[1+p]_q[2+p]_q}K(q,p,1,\beta) & \text{if } \lambda < \gamma_3\ ; \\ \frac{[2]_q[3]_q(1-\beta)}{q[1+p]_q[2+p]_q} & \text{if } \gamma_3 \leq \lambda \leq \gamma_4\ ; \\ -\frac{[2]_q[3]_q(1-\beta)}{q^2[1+p]_q[2+p]_q}K(q,p,1,\beta) & \text{if } \gamma_4 < \lambda, \end{cases}$$

where

$$K(q,p,1,\beta) = (1-\beta)[2]_q - q^2 + 2q - \frac{\lambda(1-\beta)\left([2]_q\right)^3[2+p]_q}{[1+p]_q[3]_q},$$

$$\gamma_3 = \frac{[1+p]_q[3]_q\left\{[2]_q(1-\beta) - q^2 + q\right\}}{(1-\beta)\left([2]_q\right)^3[2+p]_q},$$

and

$$\gamma_4 = \frac{[1+p]_q[3]_q\left\{[2]_q(1-\beta) - q(q-1) + 2q\right\}}{(1-\beta)\left([2]_q\right)^3[2+p]_q}.$$

Each of these results is sharp.

The coefficient bounds obtained so far are sharp but not yet phrased as simple "if and only if" coefficient conditions. In this subsection, we give equivalent formulations of the definitions of the classes $\mathcal{LS}_{q,p-1}^{\xi-1}$ and $\mathcal{LS}_{q,p-1}^{\xi-1}(\beta)$ in terms of explicit inequalities involving the Taylor coefficients of f. From the defining subordination relations, one easily checks that

$$f \in \mathcal{LS}_{q,p-1}^{\xi-1} \Leftrightarrow \left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} - \frac{1}{1-q} \right| < \frac{1}{1-q}. \tag{3.33}$$

$$f \in \mathcal{S}_{q,p-1}^{\xi-1}(\beta) \Leftrightarrow \left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} \mathcal{V}_{q,p}^{\xi-1} f\right)(z)} - \left(\beta + \frac{1-\beta}{1-q}\right) \right| < \frac{1-\beta}{1-q}. \tag{3.34}$$

We now show how these relations can be rewritten as coefficient inequalities.

Theorem 3.3.6. *We say that $f \in \mathcal{A}_p$ is a member of the family $\mathcal{LS}_{q,p-1}^{\xi-1}$ when the following condition holds:*

$$\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q! \, [p]_q!} \left([j+1]_q\right)^{\xi-1} |a_{j+p}| < 1. \tag{3.35}$$

Proof. Given that (3.35) is satisfied, it is enough to demonstrate that

$$\left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Thus,

$$\left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} - \frac{1}{1-q} \right| < \left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} - 1 \right| + \frac{q}{1-q}$$

$$= \left| \frac{\left(D_q^{(p-1)} V_{q,p}^{\xi} f\right)(z) - \left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} \right| + \frac{q}{1-q}$$

$$= \left| \frac{\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j+1]_q!} \left([j+1]_q\right)^{\xi-1} \left([j+1]_q - 1\right) a_{j+p} z^j}{[p]_q! + \sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j+1]_q!} \left([j+1]_q\right)^{\xi-1} a_{j+p} z^j} \right| + \frac{q}{1-q}$$

$$< \frac{\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q!} \left([j+1]_q\right)^{\xi-2} \left([j+1]_q - 1\right) |a_{j+p}|}{[p]_q! - \sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q!} \left([j+1]_q\right)^{\xi-2} |a_{j+p}|} + \frac{q}{1-q}.$$

Under the assumption that (3.35) is satisfied, the last expression does not exceed $\frac{1}{1-q}$, which concludes the proof. □

Example 3.3.7. *Consider $f \in \mathcal{A}_p$, defined by $f(z) = z^p + z^{p+1}$ which implies that $a_{p+1} = 1$ and $a_{j+p} = 0$, for all $j \geq 2$. Therefore, using the criterion established in Theorem 3.3.6, $f \in \mathcal{LS}_{q,p-1}^{\xi-1}$ provided that the condition below is satisfied:*

$$[1+p]_q \left([2]_q\right)^{\xi-1} < 1.$$

The following result is obtained by substituting $\xi = 1$ in Theorem 3.3.6.

Corollary 3.3.8. *A function $f \in \mathcal{A}_p$ belongs to $\mathcal{LS}^{0}_{q,p-1}$ under condition:*

$$\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q!\,[p]_q!} |a_{j+p}| < 1.$$

We now extend the same strategy to the class $\mathcal{LS}^{\xi-1}_{q,p-1}(\beta)$.

Theorem 3.3.9. *The analytic function $f \in \mathcal{A}_p$ lies in the subclass $\mathcal{LS}^{\xi-1}_{q,p-1}(\beta)$ if the following condition holds:*

$$\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q!\,[p]_q!} \left([j+1]_q\right)^{\xi-2} \left([j+1]_q - \beta\right) |a_{j+p}| < 1-\beta. \tag{3.36}$$

Proof. Once condition (3.36) is assumed to be valid, the goal reduces to proving that:

$$\left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} - \left(\beta + \frac{1-\beta}{1-q}\right) \right| < \frac{1-\beta}{1-q}.$$

□

Hence,

$$\left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} - \left(\beta + \frac{1-\beta}{1-q}\right) \right| < \left| \frac{z\left(D_q^{(p)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} \mathcal{V}_{q,p}^{\xi-1} f\right)(z)} - 1 \right| + \frac{(1-\beta)\,q}{1-q}$$

$$= \left| \frac{\left(D_q^{(p-1)} V_{q,p}^{\xi} f\right)(z) - \left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)}{\left(D_q^{(p-1)} V_{q,p}^{\xi-1} f\right)(z)} \right| + \frac{(1-\beta)\,q}{1-q}$$

$$= \left| \frac{\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j+1]_q!} \left([j+1]_q\right)^{\xi-1} \left([j+1]_q - 1\right) a_{j+p} z^j}{[p]_q! + \sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j+1]_q!} \left([j+1]_q\right)^{\xi-1} a_{j+p} z^j} \right| + \frac{(1-\beta)\,q}{1-q}$$

$$< \frac{\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q!} \left([j+1]_q\right)^{\xi-2} \left([j+1]_q - 1\right) |a_{j+p}|}{[p]_q! - \sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j]_q!} \left([j+1]_q\right)^{\xi-2} |a_{j+p}|} + \frac{(1-\beta)\,q}{1-q}.$$

Under the assumption that (3.36) holds, the final expression remains less than $\frac{1-\beta}{1-q}$, which completes the proof.

Example 3.3.10. *Let again $f \in \mathcal{A}_p$, defined by $f(z) = z^p + z^{p+1}$ so that $a_{p+1} = 1$ and $a_{j+p} = 0$, for all $j \geq 2$. We now verify whether f belongs to the class $\mathcal{LS}^{\xi-1}_{q,p-1}(\beta)$ based on the sufficient condition stated in Theorem 3.3.9. Since all terms vanish except for $j = 1$, the inequality (3.36) simplifies to*

$$[1+p]_q \left([2]_q\right)^{\xi-2} \left([2]_q - \beta\right) < 1-\beta.$$

In particular, for $\xi = 1$, the inequality reduces to

$$\frac{[1+p]_q \left([2]_q - \beta\right)}{[2]_q} < 1-\beta,$$

which coincides with the required condition, thereby confirming the validity of the example.

Corollary 3.3.11. *Let $f \in \mathcal{A}_p$. Then f is included in the subclass $\mathcal{LS}^0_{q,p-1}$ whenever the ensuing inequality is satisfied::*

$$\sum_{j=1}^{\infty} \frac{[j+p]_q!}{[j+1]_q!\,[p]_q!}\left([j+1]_q - \beta\right)|a_{j+p}| < 1. \tag{3.37}$$

Proof. Substituting $\xi = 1$ into Theorem 3.3.9 yields the result. □

3.4 Janowsky-Type q-Multivalent Classes Defined by Higher-Order q-Derivatives

The theory of univalent and multivalent functions has seen major developments through the introduction of various subclasses defined by analytic, geometric, or operator-based properties. A fruitful approach involves the use of differential operators and subordination principles to describe starlike or convex function classes. When extended to the framework of q-calculus, these ideas give rise to discrete analogues such as q -starlike or q-convex functions.

For $-1 \leq B < A \leq 1$ and $0 \leq \beta < p$, Aouf [4] defined the class $P(A, B; p; \beta)$, which represents a subclass of the family $\mathcal{A}_p$ and comprises functions represented by

$$\varrho(z) = p + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,$$

with the property that

$$\varrho(z) \prec \frac{p + [pB + (A-B)(p-\beta)]z}{1 + Bz}.$$

In the same works Aouf (see [4]) introduced the subclass $\mathcal{S}^*(A, B; p; \beta)$ of multivalent quasi-starlike functions defined by a subordination involving the derivative $zf'(z)/f(z)$ and the fractional transformation of Janowski-type. Later, authors like Srivastava et al. (see [15, 20, 16]) developed further classes using higher-order q-derivatives and generalized subordination functions.

In this part of the chapter, we extend this line of research to the q -calculus setting and fill a gap in the literature by introducing a unified framework for Janowski-type subclasses defined via higher-order q -derivatives, in both the positive- and negative-coefficient cases. More precisely, we define two new normalized q-classes within the standard family $\mathcal{A}_p$, driven by higher-order q-differential operators and Janowski subordination. For these classes we obtain sharp coefficient inequalities, structural properties such as invariance under weighted and arithmetic means, parameter monotonicity, and distortion bounds. Furthermore, we establish Goodman–Ruscheweyh–type neighborhood inclusions and study the action of fractional q-integral operators, with particular emphasis on the q-Jung–Kim–Srivastava operator. As $q \to 1^-$, all our main results reduce to known classical results in geometric function theory.

To begin, we introduce the generalized Janowski-type function

$$\phi^{[\mathfrak{n}]_q}_{A,B;\beta}(z) = \frac{[\mathfrak{n}]_q + (B[\mathfrak{n}]_q + (A-B)([\mathfrak{n}]_q - \beta))z}{1 + Bz}, \tag{3.38}$$

where $-1 \leq B < A \leq 1$ and $0 \leq \beta < [\mathfrak{n}]_q, \mathfrak{n} \in \mathbb{N}, \mathfrak{n} \leq n$.

We now define the central family of classes associated with higher-order q-derivatives.

Definition 3.4.1. *Let $p \in \mathbb{N}$, $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with $0 \le r \le p-1$, $q \in (0,1)$, $\xi \in [0,1]$, $0 \le \beta < [p-r]_q$ and $A, B \in \mathbb{R}$ with $-1 \le B < A \le 1$. A function $f \in \mathcal{A}_p$ is said to belong to the class $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ if*

$$\frac{zD_q^{(r+1)}f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!}z^{p-r} + \xi D_q^{(r)}f(z)} \prec \phi_{A,B;\beta}^{[p-r]_q}(z), \quad z \in \mathbb{U}, \tag{3.39}$$

that is, there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$\frac{zD_q^{(r+1)}f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!}z^{p-r} + \xi D_q^{(r)}f(z)} = \phi_{A,B;\beta}^{[p-r]_q}(w(z)),$$

with $\phi_{A,B;\beta}^{[p-r]_q}$ given by Equation (3.38).

Remark 3.4.2. *Because the series representations of $D_q^{(r)}f$ and $D_q^{(r+1)}f$ are absolutely and uniformly convergent on compact subsets of $\mathbb{U}$, all termwise operations are justified.*

Throughout, we assume, without repeated mention, that the following conditions hold: $z \in \mathbb{U}$, $p \in \mathbb{N}$, $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with $0 \le r \le p-1$, $q \in (0,1)$, $\xi \in [0,1]$, $0 \le \beta < [p-r]_q$, and $A, B \in \mathbb{R}$ with $-1 \le B < A \le 1$.

The class $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ reduces to several known subclasses of multivalent functions for particular choices of parameters:

- *If $q \to 1^-$, $\xi = 1$, and $r = 0$, then $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ reduces to the class $\mathcal{S}^*(A, B; p; \beta)$ introduced by Aouf (see [4]) where*

$$\frac{zf'(z)}{f(z)} \prec \frac{p + [pB + (A-B)(p-\beta)]z}{1 + Bz}.$$

- *If $B = -1$, $A = 1$, $q \to 1^-$ and $\xi = 1$, then $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ becomes the class of p-valent starlike functions of order β, $\mathcal{S}_p^*(\beta)$.*

- *If $r = 0$, $\xi = 1$, $\beta = 0$ and $q \to 1^-$, then $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ specializes to the class $\mathcal{S}_p^*(A, B)$ investigated by Hayami and Owa [10].*

- *If $p = 1$, $r = 0$, $\xi = 1$, $q \to 1^-$ then $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ reduces to the class $\mathcal{S}_p^*(A, B; \beta)$ proposed by Polatoglu et al. [21].*

- *If $p = 1$, $r = 0$, $\xi = 1$, $q \to 1^-$ and $\beta = 0$, then $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ reduces to the subclass $\mathcal{S}^*(A, B)$ considered by Janowski [12] and was further studied by Goel and Mehrok [7].*

- *If $B = -1$, $A = 1$, $q \to 1^-, p = 1$, $r = 0$, $\xi = 1$, then $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ reduces to $\mathcal{S}^*(\beta)$.*

- *If $B = -1$, $A = 1$, $q \to 1^-, p = 1$, $r = 0$, $\xi = 1$ and $\beta = 0$, then $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ reduces to $\mathcal{S}^*$.*

In addition, we introduce $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta) = \mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta) \bigcap \mathcal{T}_p$.

By assigning specific values to the parameter values, we recover several well-known subclasses of analytic functions in $\mathbb{U}$ with negative coefficients:

- *If $q \to 1^-$, $\xi = 1$ and $r = 0$, then $\mathcal{TM}_p^{(0)}(A, B; 1, \beta)$ becomes the class introduced and studied by Aouf (see [5]).*

- *If $N=-1$, $M=1$, $q \to 1^-$, $\xi=1$ and $r=0$, then $\mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$ reduces to the class $\mathcal{P}_p^*(\beta)$ proposed by Sekine and Owa (see [24]).*
- *If $N=-\alpha$, $M=\alpha$, $q \to 1^-$, $\xi=1$, $r=0$, $\beta=1$, $p=1$, then $\mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$ reduces to the class $\mathcal{D}^*(\alpha)$ investigated by Kim and Lee (see [17]).*

For $\xi=0$, we denote the class $\mathcal{TM}_{p,q}^{(r)}(A,B;0,\beta)$ by $\mathcal{TM}_{p,q}^{(r)}(A,B;\beta)$.

Finally, we recall the following lemma, which will be used repeatedly in the inclusion results.

Lemma 3.4.3 ([18])**.** *Let $-1 \le N_2 \le N_1 < A_1 \le A_2 \le 1$, then*

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Next, we derive a coefficient inequality that characterizes functions belonging to the class $\mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$. Building on this characterization, we establish sharp coefficient bounds and investigate inclusion properties of various subclasses. Furthermore, we examine the behavior of these classes under weighted and arithmetic mean operations, prove several distortion theorems, and derive related corollaries. These results provide a unified framework for analyzing geometric properties of q-multivalent functions with negative coefficients.

Theorem 3.4.4. *Suppose $f(z)=z^p-\sum_{j=1}^{\infty}|a_{j+p}|\,z^{j+p}$ is analytic in the unit disk $\mathbb{U}$. The function f belongs to the class $\mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$ if and only if*

$$\sum_{j=1}^{\infty}\frac{[p+j]_q!}{[j+p-r]_q!}\left\{\left|[p-r]_q\left(\xi-q^j\right)-[j]_q\right|+|\chi_{j,q}|\right\}|a_{j+p}|\le C, \tag{3.40}$$

where

$$\chi_{j,q} \;=\; \left(-\xi A+Bq^j\right)[p-r]_q+\xi\beta\,(A-B)+B[j]_q, \tag{3.41}$$

and

$$C \;=\; (A-B)\left([p-r]_q-\beta\right)\frac{[p]_q!}{[p-r]_q!}. \tag{3.42}$$

Proof. The subordination condition defining the class $\mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$ can be written equivalently as

$$\left|\frac{\dfrac{zD_q^{(r+1)}f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!}z^{p-r}+\xi D_q^{(r)}f(z)}-[p-r]_q}{B[p-r]_q+(A-B)\left([p-r]_q-\beta\right)-B\dfrac{zD_q^{(r+1)}f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!}z^{p-r}+\xi D_q^{(r)}f(z)}}\right|<1. \tag{3.43}$$

Using the basic-number relation

$$[j+p-r]_q=[j]_q+q^j[p-r]_q, \qquad j,p,r\in\mathbb{N}_0, \tag{3.44}$$

which follows directly from the definition $[m]_q=\frac{1-q^m}{1-q}$, $m\in\mathbb{N}_0$ and substituting the series expansion for $D_q^{(r)}f$ and $D_q^{(r+1)}f$, we obtain

$$\left|\frac{\sum_{j=1}^{\infty}\frac{[p+j]_q!}{[j+p-r]_q!}\left\{[p-r]_q\left(\xi-q^j\right)-[j]_q\right\}a_{j+p}z^{j+p-r}}{Cz^{p-r}+\sum_{j=1}^{\infty}\frac{[p+j]_q!}{[j+p-r]_q!}\chi_{j,q}a_{j+p}z^{j+p-r}}\right|<1.$$

Taking $|z| = 1$ and applying the triangle inequality to numerator and denominator, we get

$$\frac{\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} \left|[p-r]_q \left(\xi - q^j\right) - [j]_q\right| |a_{j+p}|}{C - \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} |\chi_{j,q}| \, |a_{j+p}|} < 1,$$

which leads directly to Equation (3.40), where $\chi_{j,q}$ and C are given by Equations (3.41) and (3.42).

In the reverse direction, let $f(z) = z^p - \sum_{j=1}^{\infty} a_{j+p} z^{j+p}$ belong to class $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$. So, we have

$$\left| \frac{\frac{z D_q^{(r+1)} f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!} z^{p-r} + \xi D_q^{(r)} f(z)} - [p-r]_q}{B[p-r]_q + (A-B)\left([p-r]_q - \beta\right) - B \frac{z D_q^{(r+1)} f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!} z^{p-r} + \xi D_q^{(r)} f(z)}} \right|$$

$$= \left| \frac{-\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} \left\{[j]_q + [p-r]_q \left(q^j - \xi\right)\right\} a_{j+p} z^{j+p-r}}{C z^{p-r} - \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} \left\{\left(\xi A - B q^j\right)[p-r]_q - \xi\beta\,(A-B) - B[j]_q\right\} a_{j+p} z^{j+p-r}} \right| < 1.$$

Fix $\theta \in \mathbb{R}$ and write $z = \rho e^{i\theta}$ with $0 < \rho < 1$. To pass from the complex inequality to the real-part bound, we choose z such that the transformed ratio is real, and use the inequality $\operatorname{Re} w \leq |w|$. We have

$$\operatorname{Re}\left\{ \frac{\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} \left|[p-r]_q \left(\xi - q^j\right) - [j]_q\right| |a_{j+p}| \, z^{j+p-r}}{C z^{p-r} - \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} |\chi_{j,q}| \, |a_{j+p}| \, z^{j+p-r}} \right\} < 1.$$

The limit as $\rho \to 1^-$ is then justified by dominated convergence, owing to the absolute and uniform convergence of the underlying series. So, we obtain

$$\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} \left\{\left|[p-r]_q \left(\xi - q^j\right) - [j]_q\right| + |\chi_{j,q}|\right\} |a_{j+p}| \leq C,$$

and the proof is now complete. □

Remark 3.4.5. *Since the class $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$ is a subset of $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$ it suffices for the function $f(z) = z^p + \sum_{j=1}^{\infty} a_{j+p} z^{j+p}$ to satisfy Equation (3.40) of the previous theorem in order to be a member of $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$.*

The theorem establishes a necessary and sufficient condition for a function with negative coefficients to belong to $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$, expressed entirely in terms of the magnitudes of its coefficients. The proof proceeds by rewriting the defining subordination into an equivalent disk condition, inserting the q-derivative expansions, and applying the triangle inequality on the unit circle. The converse direction is obtained by reversing this argument and using the dominated convergence theorem to pass to the radial limit.

The coefficient characterization plays a central role in that follows. In particular, it allows one to identify extremal functions and to derive sharp individual coefficient estimates, as shown next.

In order to illustrate the definition of the class $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$, we provide an explicit example for specific parameter choices.

Example 3.4.6. *Let $p = 2$, $r = 0$, $q = \frac{1}{2}$, $\xi = \frac{1}{2}$, $A = 1$, $B = 0$, $\beta = \frac{1}{2}$. We have*

$$[1]_q = 1, \quad [2]_q = \tfrac{3}{2}, \quad [3]_q = \tfrac{7}{4}, \quad [2]_q! = \tfrac{3}{2}.$$

Hence $C = 1 \cdot \left(\frac{3}{2} - \frac{1}{2}\right) \cdot 1 = 1$, *where* C *is given by Equation (3.42). From Theorem 3.4.4, the extremal function corresponding to* $j = 1$ *is*

$$f(z) = z^2 - \frac{C}{K} z^3,$$

where

$$\begin{aligned} K &= \frac{[3]_q!}{[3]_q!} \left(|1 + [2]_q(q - \xi)| + |(-\xi A + Bq)[2]_q + \xi\beta(A - B) + B| \right) \\ &= \tfrac{7}{4} \left(|1 + \tfrac{3}{2}(0)| + |-\tfrac{1}{2} + \tfrac{1}{4}| \right) = \tfrac{35}{16}. \end{aligned}$$

Therefore,

$$f(z) = z^2 - \frac{16}{35} z^3$$

belongs to the class $\mathcal{TM}^{(0)}_{2,1/2}(1, 0; \frac{1}{2}, \frac{1}{2})$ *and satisfies Equation (3.40) exactly.*

Within the framework of $\mathcal{A}_p$, we now focus on a subclass determined by the signs of the coefficients.We denote by $\mathcal{T}_p$ the subclass of $\mathcal{A}_p$ consisting of all functions with negative coefficients, that is,

Corollary 3.4.7. *Let* $f(z)$ *be given by Equation*

$$f(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p}|\, z^{j+p}. \tag{3.45}$$

If $f(z) \in \mathcal{TM}^{(r)}_{p,q}(A, B; \xi, \beta)$, *then for every* $j \geq 1$ *the coefficient bound*

$$|a_{j+p}| \leq \frac{(A - B)\left([p - r]_q - \beta\right)[p]_q![j + p - r]_q!}{[p + j]_q![p - r]_q! \left\{ |[p - r]_q\,(\xi - q^j) - [j]_q| + |\chi_{j,q}| \right\}},$$

holds, where $\chi_{j,q} = (-\xi A + Bq^j)\,[p - r]_q + \xi\beta\,(A - B) + B[j]_q$. *This estimate is sharp in each coordinate: for every fixed* j, *equality is attained by the extremal function*

$$f(z) = z^p - \sum_{j=1}^{\infty} \frac{(A - B)\left([p - r]_q - \beta\right)[p]_q![j + p - r]_q!}{[p + j]_q![p - r]_q! \left\{ |[p - r]_q\,(\xi - q^j) - [j]_q| + |\chi_{j,q}| \right\}} z^{j+p}.$$

For $q \to 1^-$, $\xi = 1$ and $r = 0$ in Theorem 3.4.4, we obtain the following result previously obtained by Aouf (see [5])

Corollary 3.4.8. *The function* $f(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p}|\, z^{j+p}$ *belongs to the class* $\mathcal{TM}^{(0)}_{p}(A, B; 1, \beta)$ *if and only if*

$$\sum_{j=1}^{\infty} \left(j + |(-A + B)\,p + \beta\,(A - B) + Bj| \right) |a_{j+p}| \leq (A - B)\,(p - \beta)\,.$$

For $\xi = 0$ in Theorem 3.4.4, we obtain the following result:

Corollary 3.4.9. *Let* $f(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p}|\, z^{j+p}$ *analytic in the unit disk* $\mathbb{U}$. *The function* f *belongs to the class* $\mathcal{TM}^{(r)}_{p,q}(A, B; \beta)$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{[p + j]_q!}{[j + p - r - 1]_q!} |a_{j+p}| \leq \frac{C}{1 + B}, \tag{3.46}$$

where C *is given by Equation (3.42) and* $-1 < B < A \leq 1$.

Theorem 3.4.10. *Let the function* $f(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p}| z^{j+p}$ *belong to the class* $\mathcal{TM}_p^{(r)}(A, B; \beta)$, *where* $-1 < B < A \leq 1$. *Then*

$$\sum_{j=1}^{\infty} |a_{j+p}| \leq \frac{(A-B)\left([p-r]_q - \beta\right)}{(B+1)\,[p+1]_q}. \tag{3.47}$$

Proof. Applying Corollary 3.4.9, Equation (3.46) yields

$$\begin{aligned} &(B+1)\,\frac{[p+1]_q!}{[p-r]_q!}\sum_{j=1}^{\infty} |a_{j+p}| \\ \leq\ & \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!}\,(B+1)\,|a_{j+p}| \\ \leq\ & (A-B)\left([p-r]_q - \beta\right)\frac{[p]_q!}{[p-r]_q!}, \end{aligned}$$

hence Equation (3.47) follows immediately. □

With the coefficient characterization in hand, we can now analyze the structural behavior of $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$ under basic operations. In this subsection we show that the class is closed under certain convex combinations (weighted means and arithmetic means) and that it enjoys a natural monotonicity with respect to the parameters A and β.

The next theorem in this section shows that if two functions belong to $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$, then any real convex combination of them (with parameter $k \in [-1, 1]$) also lies in the same class. The proof is straightforward once we rewrite the new function's coefficients as linear combinations of the original ones and apply the coefficient inequality together with the non-negativity of the weights. This property reveals that $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$ is convex (in the sense of coefficient sequences).

Theorem 3.4.11. *Let*

$$f_1(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p}|\, z^{j+p}, \qquad f_2(z) = z^p - \sum_{j=1}^{\infty} |b_{j+p}|\, z^{j+p},$$

be two functions in the class $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$. *For any* $k \in [-1, 1]$, *define*

$$F_k(z) = \frac{1-k}{2}\, f_1(z) + \frac{1+k}{2}\, f_2(z).$$

Then $F_k \in \mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$.

Proof. For ease of notation, set:

$$W_j = \frac{[p+j]_q!}{[j+p-r]_q!}\,\left\{\left|[p-r]_q\left(\xi - q^j\right) - [j]_q\right| + |\chi_{j,q}|\right\} \geq 0, \tag{3.48}$$

where $\chi_{j,q}$ is given by Equation (3.41) and C is given by Equation (3.42). By the coefficient characterization Theorem 3.4.4, the functions $f_1, f_2 \in \mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$ if and only if

$$\sum_{j=1}^{\infty} W_j\, |a_{j+p}| \leq C, \ \sum_{j=1}^{\infty} W_j\, |b_{j+p}| \leq C. \tag{3.49}$$

In terms of its series representation, F_k takes the form:

$$F_k(z) = z^p - \sum_{j=1}^{\infty} c_{j+p}\, z^{j+p}, \text{ where } c_{j+p} = \frac{1-k}{2}\,|a_{j+p}| + \frac{1+k}{2}\,|b_{j+p}|.$$

Since $k \in [-1, 1]$, we have $\frac{1-k}{2} \geq 0$ and $\frac{1+k}{2} \geq 0$, hence $c_{j+p} \geq 0$ and the required sign structure of the coefficients is preserved.

Using linearity and Equation (3.49),

$$\begin{aligned}\sum_{j=1}^{\infty} W_j\, c_{j+p} &= \frac{1-k}{2} \sum_{j=1}^{\infty} W_j\, |a_{j+p}| + \frac{1+k}{2} \sum_{j=1}^{\infty} W_j\, |b_{j+p}| \\ &\leq \frac{1-k}{2}\, C + \frac{1+k}{2}\, C = C.\end{aligned}$$

Therefore the characterization Equation (3.40) holds for the coefficients of F_k, and thus $F_k \in \mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$. This proves that the class $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$ is closed under the weighted mean. □

The following result extends this idea to arithmetic means of finitely many functions from the class. Writing the mean as a function with coefficients equal to the arithmetic averages of the original coefficients, and summing the inequalities provided by Theorem 4.2.2.1., yields the desired result. Thus, the class $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$ is stable with respect to averaging procedures, which is a natural and desirable structural property.

Theorem 3.4.12. *Let*

$$f_l(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p,l}|\, z^{j+p}, \qquad l = 1, 2, \ldots, m,$$

be functions in the class $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$. *Then their arithmetic mean*

$$G(z) = \frac{1}{m} \sum_{l=1}^{m} f_l(z),$$

also belongs to $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$.

Proof. The function G can be expressed in the same negative-coefficients series representation:

$$G(z) = z^p - \sum_{j=1}^{\infty} c_{j+p}\, z^{j+p}, \qquad c_{j+p} = \frac{1}{m} \sum_{l=1}^{m} |a_{j+p,l}|.$$

Clearly $c_{j+p} \geq 0$, so the coefficient pattern required by the subclass is preserved.

In view of Equations (3.48) and (3.42) and by Theorem 3.4.4, a function $f(z) = z^p - \sum_{j=1}^{\infty} |c_{j+p}|\, z^{j+p}$ belongs to $\mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$ if and only if

$$\sum_{j=1}^{\infty} W_j\, |c_{j+p}| \;\leq\; C. \tag{3.50}$$

Since each $f_l \in \mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$, we have for every $l = 1, \ldots, m$,

$$\sum_{j=1}^{\infty} W_j\, |a_{j+p,l}| \;\leq\; C.$$

Using linearity and the definition of c_{j+p},

$$\begin{aligned}\sum_{j=1}^{\infty} W_j\, c_{j+p} &= \sum_{j=1}^{\infty} W_j\, \frac{1}{m} \sum_{l=1}^{m} |a_{j+p,l}| \\ &= \frac{1}{m} \sum_{l=1}^{m} \sum_{j=1}^{\infty} W_j\, |a_{j+p,l}| \\ &\leq \frac{1}{m} \sum_{l=1}^{m} C = C.\end{aligned}$$

Thus Equation (3.50) holds for G, and by Theorem 3.4.4 we conclude $G \in \mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$. This proves closure under arithmetic means. □

The next theorem establishes a monotonicity property with respect to the parameters.

Theorem 3.4.13. *Let $0 \leq \beta_2 \leq \beta_1 < [p-r]_q$ and $-1 \leq B < A_1 \leq A_2 \leq 1$. Then*

$$\mathcal{TM}_{p,q}^{(r)}(A_1, B; \xi, \beta_1) \subset \mathcal{TM}_{p,q}^{(r)}(A_2, B; \xi, \beta_2).$$

Proof. Let $f \in \mathcal{TM}_{p,q}^{(r)}(A_1, B; \xi, \beta_1)$. From Equation (3.39), we have

$$\frac{zD_q^{(r+1)} f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!} z^{p-r} + \xi D_q^{(r)} f(z)} \prec \frac{[p-r]_q + (B[p-r]_q + (A_1 - B)\,([p-r]_q - \beta_1))z}{1+Bz}.$$

From $0 \leq \beta_2 \leq \beta_1 < [p-r]_q$ and $-1 \leq B < A_1 \leq A_2 \leq 1$, we obtain

$$-1 \leq B + \frac{(A_1 - B)\,([p-r]_q - \beta_1)}{[p-r]_q} \leq B + \frac{(A_2 - B)\,([p-r]_q - \beta_2)}{[p-r]_q} \leq 1.$$

The middle inequality in the chain follows because the expression $B + \frac{(A-B)([p-r]_q - \beta)}{[p-r]_q}$ increases with M and decreases with β.

In view of Lemma 3.4.3, we get

$$\frac{zD_q^{(r+1)} f(z)}{(1-\xi)\frac{[p]_q!}{[p-r]_q!} z^{p-r} + \xi D_q^{(r)} f(z)} \prec \frac{[p-r]_q + (B[p-r]_q + (A_2 - B)\,([p-r]_q - \beta_2))z}{1+Bz}.$$

Hence, $f \in \mathcal{TM}_{p,q}^{(r)}(A_2, B; \xi, \beta_2)$. □

3.5 Applications to Distortion, Neighborhoods, and Fractional q-Integral Operators

We, now, turn to distorsion estimates for functions in $\mathcal{TM}_{p,\beta}^{(r)}\,(A, B, \xi, \beta)$

Theorem 3.5.1. *If $f \in \mathcal{TM}_{p,q}^{(r)}(A, B; \xi, \beta)$, then for $|z| = \rho$, $0 < \rho < 1$,*

$$\rho^p - \frac{C}{W_1}\rho^{p+1} \leq |f(z)| \leq \rho^p + \frac{C}{W_1}\rho^{p+1}, \tag{3.51}$$

where C is given in Equation (3.42) and

$$W_1 = \frac{[p+1]_q!}{[p+1-r]_q!} \left\{ |1 + [p-r]_q(q-\xi)| + |(-\xi A + Bq)\,[p-r]_q + \xi\beta\,(A-B) + B| \right\}. \tag{3.52}$$

The bounds in Equation (3.51) are sharp and are achieved by the function $f(z)$ defined by

$$f(z) = z^p - \frac{C}{W_1} z^{p+1}, \tag{3.53}$$

at $z = \rho$ and $z = \rho \exp\left((2k+1)\,\pi i\right)$.

Proof. By Theorem 3.4.4, one readily gets

$$W_1 \sum_{j=1}^{\infty} |a_{j+p}| \le \sum_{j=1}^{\infty} W_j |a_{j+p}| \le C,$$

and thus

$$|f(z)| \le \rho^p + \sum_{j=1}^{\infty} |a_{j+p}| \rho^{p+j} \le \rho^p + \rho^{p+1} \sum_{j=1}^{\infty} |a_{j+p}| \le \rho^p + \frac{C}{W_1} \rho^{p+1}.$$

A similar estimate yields the lower bound.

$$|f(z)| \ge \rho^p - \sum_{j=1}^{\infty} |a_{j+p}| \rho^{p+j} \ge \rho^p - \rho^{p+1} \sum_{j=1}^{\infty} |a_{j+p}| \ge \rho^p - \frac{C}{W_1} \rho^{p+1}.$$

The extremal function $f(z) = z^p - \frac{C}{W_1} z^{p+1}$ attains equality in (3.51) at $z = \rho$ (for the upper bound) and $z = \rho \exp^{(2k+1)\pi i}$ (for the lower bound), since in these cases $|f(\rho)| = \rho^p - \frac{C}{W_1}\rho^{p+1}$. This confirms the sharpness of both bounds.

Hence, the proof is finished. □

For $q \to 1^-$, $\xi = 1$ and $r = 0$, we obtain the following result previously obtained by Aouf (see [5]).

Corollary 3.5.2. *If $f \in \mathcal{TM}_p^{(0)}(A, B; 1, \beta)$, then for $|z| = \rho$, $0 < \rho < 1$,*

$$\rho^p - \frac{(A-B)\,(p-\beta)}{1 + |B + (B-A)\,(p-\beta)|} \rho^{p+1} \le |f(z)| \le \rho^p + \frac{(A-B)\,(n-\beta)}{1 + |B + (B-A)\,(p-\beta)|} \rho^{p+1}.$$

For $\xi = 0$, we get:

Corollary 3.5.3. *If $f \in \mathcal{TM}_{p,q}^{(r)}(A, B; \beta)$ and $N \ge 0$, then for $|z| = \rho$, $0 < \rho < 1$,*

$$|f(z)| \ge \rho^p - \frac{C}{w_1} \rho^{p+1}, \qquad |f(z)| \le \rho^p + \frac{C}{w_1} \rho^{p+1}, \tag{3.54}$$

where C is given in Equation (3.42) and

$$w_1 = \frac{[p+1]_q!}{[p+1-r]_q!} \left\{ 1 + [p-r]_q\, q + Bq\ [p-r]_q + B \right\}. \tag{3.55}$$

The bounds in Equation (3.54) are sharp and are achieved by

$$f(z) = z^p - \frac{C}{w_1}\, z^{p+1} \tag{3.56}$$

at $z = \rho$ and $z = \rho e^{(2k+1)\pi i}$.

The proof of the next result is analogous to that of Theorem 3.5.1, and is therefore omitted.

Theorem 3.5.4. *If $f \in \mathcal{TM}^{(r)}_{p,q}(A, B; \xi, \beta)$, then for $|z| = \rho$, $0 < \rho < 1$,*

$$|D_q f(z)| \geq [p]_q \rho^{p-1} - [p+1]_q \frac{C}{W_1} \rho^p, \tag{3.57}$$

and

$$|D_q f(z)| \leq [p]_q \rho^{p-1} + [p+1]_q \frac{C}{W_1} \rho^p, \tag{3.58}$$

where C and W_1 are given by Equations (3.42) and (3.52). The bounds in Equations (3.57) and (3.58) are sharp and are achieved by the function specified by Equation (3.51).

For $q \to 1^-$, $\xi = 1$ and $r = 0$, we obtain the following result previously obtained by Aouf (see [5]).

Corollary 3.5.5. *If $f \in \mathcal{TM}^{(0)}_p(A, B; 1, \beta)$, then for $|z| = \rho$, $0 < \rho < 1$,*

$$p\,^{p-1} - \frac{(p+1)(A-B)(p-\beta)}{1 + |B + (B-A)(p-\beta)|}\rho^p \leq \left|f^{'}(z)\right| \leq p\rho^{p-1} + \frac{(p+1)(A-B)(p-\beta)}{1 + |B + (B-A)(p-\beta)|}\rho^p.$$

For $\xi = 0$, we get:

Corollary 3.5.6. *If $f \in \mathcal{TM}^{(r)}_{p,q}(A, B; \beta)$ and $B \geq 0$, then for $|z| = \rho$, $0 < \rho < 1$,*

$$|D_q f(z)| \geq [p]_q \rho^{p-1} - [p+1]_q \frac{C}{w_1} \rho^p, \tag{3.59}$$

$$|D_q f(z)| \leq [p]_q \rho^{p-1} + [p+1]_q \frac{C}{w_1} \rho^p, \tag{3.60}$$

where C is given in Equation (3.42) and w_1 is given in Equation (3.55). The bounds are sharp and are achieved by Equation (3.56).

Example 3.5.7. *Let's take*

$$p = 1, \quad r = 0, \quad q = 0.7, \quad \xi = \tfrac{1}{2}, \quad M = 0.8, \quad N = -0.2, \quad \beta = 0.3.$$

Then $[p-r]_q = [1]_q = 1$, $[p]_q! = [1]_q! = 1$, hence

$$C = (0.8 - (-0.2))(1 - 0.3) = 0.7,$$

where C is given by (3.42). We also record the basic q-numbers (with $q = 0.7$):

$$[1]_q = \frac{1 - 0.7}{1 - 0.7} = 1, \qquad [2]_q = \frac{1 - 0.7^2}{1 - 0.7} = \frac{1 - 0.49}{0.3} = 1.7.$$

We give now coefficient bounds from Theorem 3.4.4 (for $j = 1, 2$): For $p = 1, r = 0$, Theorem 3.4.4 gives

$$|a_{j+1}| \leq \frac{(A-B)([1]_q - \beta)[1]_q!\,[j+1]_q!}{[1+j]_q!\,[1]_q!\ \left(|[1]_q(\xi - q^j) - [j]_q| + |\chi_{j,q}|\right)} = \frac{C}{|(\xi - q^j) - [j]_q| + |\chi_{j,q}|},$$

with $\chi_{j,q}$ given by Equation (3.41). For $j = 1$: $[j]_q = 1$, $q^j = 0.7$.

$$|(\xi - q^1) - [1]_q| = |(0.5 - 0.7) - 1| = 1.2,$$

$$\chi_{1,q} = (-\tfrac{1}{2}\cdot 0.8 + (-0.2)\cdot 0.7)\cdot 1 + \tfrac{1}{2}\cdot 0.3\cdot (1.0) + (-0.2)\cdot 1 = -0.59, \quad |\chi_{1,q}| = 0.59.$$

Hence,

$$|a_2| \;\le\; \frac{0.7}{1.2+0.59} \;\approx\; 0.391.$$

For $j=2$*:* $[j]_q = [2]_q = 1.7,\; q^j = 0.49.$

$$|(\xi - q^2) - [2]_q| = |(0.5-0.49) - 1.7| = 1.69,$$

$$\chi_{2,q} = (-0.5\cdot 0.8 + (-0.2)\cdot 0.49)\cdot 1 + 0.5\cdot 0.3\cdot (1.0) + (-0.2)\cdot 1.7 = -0.688, \quad |\chi_{2,q}| = 0.688.$$

Hence,

$$|a_3| \;\le\; \frac{0.7}{1.69+0.688} \;\approx\; 0.294.$$

Distortion estimate at $\rho = 0.8$*. For* $p=1, r=0$*,*

$$\frac{[2]_q!}{[2]_q!} = 1, \quad |1+[1]_q(q-\xi)| = |1+(0.7-0.5)| = 1.2,$$

Thus,

$$W_1 = 1.2 + 0.59 = 1.79, \qquad \frac{C}{W_1} = \frac{0.7}{1.79} \approx 0.391,$$

where W_1 *and* C *are given by Equations (3.41) and (3.42).*

Therefore, for $|z| = \rho = 0.8$ *and* $p=1$*,*

$$|f(z)| \;\in\; \left[\rho - \tfrac{C}{W_1}\rho^2,\; \rho + \tfrac{C}{W_1}\rho^2\right] \;=\; [0.8 - 0.391\cdot 0.64, \;\; 0.8 + 0.391\cdot 0.64] \;\approx [\,0.550,\; 1.050\,].$$

Note The distortion theorem requires $\xi \le q$ *(satisfied since* $0.5 \le 0.7$*) and* $(-\xi A + Bq)[p-r]_q + \xi\beta(A-B) + B \ge 0$*. For this parameter choice, the last quantity equals* $-0.59 < 0$*; if one chooses* $N \ge 0$*, the hypothesis is strictly satisfied.*

In the next step, we investigate neighborhood inclusion properties for the function class $\mathcal{TM}_{p,q}^{(r)}(A,B;\beta)$. Following the classical approach initiated by Goodman and Ruscheweyh, we introduce a suitable notion of μ-neighborhood for functions in $\mathcal{T}_p$ and establish sufficient conditions under which such neighborhoods are contained in the class $\mathcal{TM}_{p,q}^{(r)}(A,B;\beta)$. These results provide a natural extension of earlier neighborhood theorems to the setting of q-calculus and multivalent analytic functions with negative coefficients.

In line with the classical approach of Goodman [8] and Ruscheweyh [22] (cf. [2]), we introduce the μ-neighborhood of a function $f(z) \in \mathcal{T}_p$ by

$$\mathcal{N}_\mu(f;h) = \left\{ h(z) = z^p - \sum_{j=1}^{\infty} |d_{j+p}|\, z^{j+p} \in \mathcal{T}_p : \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!}\, ||a_{j+p}| - |d_{j+p}|| \le \mu \right\}, \tag{3.61}$$

with $\mu > 0$.

Theorem 3.5.8. *Let* $f \in \mathcal{TM}_{p,q}^{(r+1)}(A,B;\beta)$*. Then* $\mathcal{N}_\mu(f;h) \subset \mathcal{TM}_{p,q}^{(r)}(A,B;\beta)$*, where*

$$\mu = \frac{(A-B)\left([p-r]_q - \beta\right)\left([p-r]_q - 1\right)[p]_q!}{(1+B)\,[p-r]_q\,[p-r]_q!}, \tag{3.62}$$

with $-1 < B < A \le 1$*.*

Proof. If $f \in \mathcal{TM}_{p,q}^{(r+1)}(A, B; \beta)$, Corollary 3.4.9 implies

$$\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-2]_q!} |a_{j+p}| \leq \frac{C}{1+B},$$

where C is given by Equation (3.42).

Hence

$$\sum_{j=1}^{\infty} \frac{[p+j]_q! \ [j+p-r-1]_q}{[j+p-r-1]_q!} |a_{j+p}| \leq \frac{C}{1+B},$$

so that

$$[p-r]_q \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!} |a_{j+p}| \leq \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-2]_q!} |a_{j+p}| \leq \frac{C}{1+B},$$

which is equivalent to

$$\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!} |a_{j+p}| \leq \frac{C}{(1+B)\,[p-r]_q}. \tag{3.63}$$

Now take $h(z) = z^p - \sum_{j=1}^{\infty} |d_{j+p}| \, z^{j+p} \in \mathcal{N}_\mu(f; h)$, where μ is given in Equation (3.62). From Equation (3.61) we have

$$\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!} \left||a_{j+p}| - |d_{j+p}|\right| \leq \mu, \qquad \mu > 0. \tag{3.64}$$

From Equations (3.63) and (3.64) it follows that

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!} |d_{j+p}| &\leq \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!} |a_{j+p}| \\
&\quad + \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r-1]_q!} \left||a_{j+p}| - |d_{j+p}|\right| \\
&\leq \frac{C}{(1+B)\,[p-r]_q} + \mu = \frac{(A-B)([p-r]_q - \beta)\,[p]_q!}{(1+B)\,[p-r]_q\,[p-r]_q!} \\
&\quad + \frac{(A-B)([p-r]_q - \beta)([p-r]_q - 1)\,[p]_q!}{(1+B)\,[p-r]_q\,[p-r]_q!} \\
&= \frac{(A-B)([p-r]_q - \beta)\,[p]_q!}{(1+B)\,[p-r]_q!} = \frac{C}{1+B}.
\end{aligned}$$

Therefore, applying Corollary 3.4.9 again, we conclude that $h(z) \in \mathcal{TM}_{p,q}^{(r)}(A, B; \beta)$. $\square$

For $\beta = 0$, we obtain

Corollary 3.5.9. *Let $f \in \mathcal{TM}_{p,q}^{(r+1)}(A, B; 0)$. Then $\mathcal{N}_\mu(f; h) \subset \mathcal{TM}_{p,q}^{(r)}(A, B; 0)$, where*

$$\mu = \frac{(A-B) - ([p-r]_q - 1)\,[p]_q!}{(1+B)\,[p-r]_q!},$$

with $-1 < B < A \leq 1$.

In this section, we investigate how the newly defined function classes behave under the action of fractional q-integral operators, with a particular focus on the q-Jung–Kim–Srivastava operator.

By employing the fractional q-integral operator for $f \in \mathcal{T}_p$, the q-Jung–Kim–Srivastava integral operator is defined as follows (see [14]):

$$\begin{aligned}\mathcal{JKS}_{q,p}^{-\alpha} f(z) &= \frac{\Gamma_q(c+\alpha+p)}{\Gamma_q(c+p)} z^{1-c-\alpha} D_{q,z}^{-\alpha}\left(z^{c-1} f(z)\right) \\ &= z^p - \sum_{j=1}^{\infty} \frac{\Gamma_q(c+\alpha+p)\Gamma_q(c+j+p)}{\Gamma_q(c+p)\Gamma_q(c+j+p+\alpha)} |a_{j+p}|\, z^{j+p}, \qquad (3.65)\end{aligned}$$

where $c > -p$, $\alpha > 0$.

Remark 3.5.10. *For $p = 1$ and $q \to 1^-$ in Equation (3.65), we obtain the Jung–Kim–Srivastava integral operator defined in [13] as*

$$\mathcal{JKS}^{-\alpha} f(z) = z - \sum_{j=1}^{\infty} \frac{\Gamma(c+\alpha+1)\Gamma(c+j+1)}{\Gamma(c+1)\Gamma(c+j+1+\alpha)} |a_{j+1}|\, z^{j+1}.$$

Theorem 3.5.11. *If $f \in \mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$ and $c > -p$, $\alpha > 0$, then $\mathcal{JKS}_{q,p}^{-\alpha} f \in \mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$.*

Proof. Let $f(z) = z^p - \sum_{j=1}^{\infty} |a_{j+p}| z^{j+p}$ and

$$\mathcal{JKS}_{q,p}^{-\alpha} f(z) = z^p - \sum_{j=1}^{\infty} \frac{\Gamma_q(c+\alpha+p)\Gamma_q(c+j+p)}{\Gamma_q(c+p)\Gamma_q(c+j+p+\alpha)} |a_{j+p}|\, z^{j+p}.$$

$\mathcal{JKS}_{q,p}^{-\alpha} f \in \mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$ if

$$A = \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} \left\{ \left| [p-r]_q \left(\xi - q^j\right) - [j]_q \right| + |\chi_{j,q}| \right\} \frac{\Gamma_q(c+\alpha+p)\Gamma_q(c+j+p)}{\Gamma_q(c+p)\Gamma_q(c+j+p+\alpha)} |a_{j+p}| \le C,$$

where $\chi_{j,q}$ and C are given by Equations (3.41) and (3.42).

Since $c > -p$ and $\alpha > 0$, all arguments of Γ_q are positives. Using that Γ_q is log-convex on $(0,\infty)$ for $q \in (0,1)$, the function $x \mapsto \frac{\Gamma_q(x+\alpha)}{\Gamma_q(x)}$ is increasing; hence, for every $j \in \mathbb{N}$,

$$\frac{\Gamma_q(c+\alpha+p)}{\Gamma_q(c+p)} \le \frac{\Gamma_q(c+j+p+\alpha)}{\Gamma_q(c+j+p)}$$

implies that

$$\frac{\Gamma_q(c+\alpha+p)\,\Gamma_q(c+j+p)}{\Gamma_q(c+p)\,\Gamma_q(c+j+p+\alpha)} \le 1.$$

So, it is evident that

$$A \le \sum_{j=1}^{\infty} \frac{[p+j]_q!}{[j+p-r]_q!} \left\{ \left| [p-r]_q \left(\xi - q^j\right) - [j]_q \right| + |\chi_{j,q}| \right\} |a_{j+p}|.$$

By Equation (3.40), we have

$$A \le C,$$

hence $\mathcal{JKS}_{q,p}^{-\alpha} f \in \mathcal{TM}_{p,q}^{(r)}(A,B;\xi,\beta)$. □

Corollary 3.5.12. *If $f \in \mathcal{TM}_{p,q}^{(r)}(A,B;\beta)$ and $c > -p$, $0 \le \alpha < 1$, then $\mathcal{JKS}_{q,p}^{-\alpha} f \in \mathcal{TM}_{p,q}^{(r)}(A,B;\beta)$.*

3.6 Remarks

In this chapter, we have explored a unified approach to analytic function classes defined by higher-order q-derivatives, highlighting the interplay between q-calculus, subordination theory, and geometric function properties. Two complementary frameworks were examined: the operator-induced normalized q-classes generated by the generalized differential operator $V_{q,p}^{\beta}$, and the Janowski-type q-multivalent classes $\mathcal{M}_{p,q}^{(r)}(A, B; \xi, \beta)$, together with their negative-coefficient variants. Both constructions reveal deep connections between q-difference operators and fundamental geometric characteristics such as starlikeness, convexity, growth estimates, and coefficient behavior.

A central theme emerging throughout the chapter is the pivotal role of subordination. Subordination provides not only the structural foundation for defining these q-deformed classes but also the principal method for establishing inclusion relationships, deriving Fekete–Szegő–type coefficient inequalities, and obtaining sharp bounds involving higher-order q-derivatives. The results show that many classical geometric principles remain stable under q-deformation and that the q-framework enriches the available analytic tools in a systematic way.

The chapter also demonstrates that classes defined through higher-order q-derivatives are stable under several fundamental operations, including weighted and arithmetic means, as well as fractional q-integral transformations such as the q-Jung–Kim–Srivastava operator. These structural properties highlight the robustness of the newly defined classes and suggest natural avenues of generalization. Likewise, the coefficient characterization theorems, distortion estimates, and Goodman–Ruscheweyh–type neighborhood inclusions establish a strong parallel between classical function theory and its q-analogue.

The results presented here open several promising directions for future research. Potential developments include studying extremal problems for the introduced classes, deriving sharp radii results (radii of starlikeness, convexity, and close-to-convexity), exploring q-analogues of differential subordinations involving other special functions, and examining the limit transitions as $q \to 1^-$, to refine or extend known classical results. Further extensions may incorporate fractional q-calculus, operator factorizations, and integral transforms that remain compatible with q-deformations.

Overall, this chapter demonstrates that higher-order q-differential operators provide a rich and versatile framework for generating and analyzing normalized function classes, yielding a coherent and unifying perspective that bridges classical geometric function theory with modern q-calculus.

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Chapter 4

Symmetric q-Differential Operators and Janowsky-Type Multivalent Function Classes

Contents

In this chapter (see [3, 4])we investigate a unified symmetric q -differential operator acting on the class $\mathcal{A}_p$ of p-valent analytic functions in the unit disk $\mathbb{U}$. The iterated operator $\widetilde{V}^{\beta}_{\tau,q,p}$ depending on the parameters $\tau \geq 0$, $0 < q < 1$, $\beta \geq 0$, $p \in \mathbb{N}$, simultaneously extends several well-known differential operators in geometric function theory: in particular, in the limiting case $q \to 1^-$, with $p = 1$ it reduces to the Al-Oboudi operator, while for $q \to 1^-$, with $p = 1$ and $\tau = 1$ it collapses to the classical Salagean operator. This provides a natural q-deformation framework in which classical results can be embedded and generalized.

Within this setting we introduce three Janowski-type subclasses of multivalent symmetric q-starlike functions associated with $\widetilde{V}^{\beta}_{\tau,q,p}$, together with a Janowski-type family $\widetilde{\mathcal{J}}^{\beta}_{q}(B, A, \tau, \lambda, p)$, defined through subordination conditions involving the same operator. For these classes we derive coefficient inequalities, obtain basic inclusion relations, and identify several classical subclasses as limiting or special cases. A second main direction of the chapter is devoted to second-order differential subordination and superordination results for transforms of the form $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}$ and $\frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}$, where we establish sharp conditions guaranteeing subordination/superordination to prescribed univalent (typically Janowski-type) functions and determine the corresponding best dominant and best subordinant. By combining these results we obtain sandwich-type theorems which yield two-sided bounds for the images of $\mathbb{U}$ under the action of the symmetric q-operator and thereby provide a coherent geometric description of the associated function classes.

4.1 The Symmetric q-Differential Operator $V^{\beta}_{\alpha,q,p}$

Let $f \in \mathcal{A}_p$, $f(z) = z^p + \sum_{j=1}^{\infty} a_{j+p} z^{j+p}$. We now introduce the following symmetric q-differential operator $\widetilde{V}_{\tau,q,p} : \mathcal{A}_p \to \mathcal{A}_p$, defined by:

$$\widetilde{V}_{\tau,q,p}f(z) = \left(1 - \tau\widetilde{[p]}_q\right) f(z) + \tau z \widetilde{D}_q f(z) = \tag{4.1}$$

$$= z^p + \sum_{j=1}^{\infty} \left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right) a_{j+p} z^{j+p}, \tag{4.2}$$

where $\tau \geq 0$.

This operator can be iterated in a natural way. For $\beta \in \mathbb{N}_0$, we first set

$$\widetilde{V}^{0}_{\tau,q,p}f(z) = f(z),$$

$$\widetilde{V}^{1}_{\tau,q,p}f(z) = \widetilde{V}_{\tau,q,p}f(z),$$

$$\widetilde{V}^{2}_{\tau,q,p}f(z) = \widetilde{V}_{\tau,q,p}(\widetilde{V}_{\tau,q,p}f(z)) = z^p + \sum_{j=1}^{\infty} \left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^2 a_{j+p} z^{j+p}.$$

By induction, we arrive at the general formula:

$$\widetilde{V}^{\beta}_{\tau,q,p}f(z) = \widetilde{V}_{\tau,q,p}(\widetilde{V}^{\beta-1}_{\tau,q,p}f(z)) = z^p + \sum_{j=1}^{\infty} \left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta} a_{j+p} z^{j+p}. \tag{4.3}$$

Remark 4.1.1. *In the limiting case $q \to 1^-$, $p = 1$, the operator described in (4.3) becomes the standard Al Oboudi's differential operator [11] in the classical framework. As*

$q \to 1^-$, $p = 1$ and $\tau = 1$, the operator given in (4.3) collapses to the traditional Salagean's differential operator [17] in the classical case.

The key motivation for introducing the operator $\widetilde{V}^{\beta}_{\tau,q,p}f(z)$ lies in its ability to generalize classical differential operator within the framework of symmetric q-calculus. By incorporating the parameters $\tau, \beta,\ q$ and p, the operator allows for a flexible treatment of multivalent analytic functions and unifies various known cases. Its structure makes it particularly suitable for studying function classes associated with Janowski-type conditions and for establishing differential subordination results with sharp bounds. Applying the symmetric q-derivative of $\widetilde{V}^{\beta}_{\tau,q,p}$ yields

$$\begin{aligned}\widetilde{D}_q \widetilde{V}^{\beta}_{\tau,q,p}f(z) &= \frac{\widetilde{V}^{\beta}_{\tau,q,p}f(qz) - \widetilde{V}^{\beta}_{\tau,q,p}f(q^{-1}z)}{z\left(q-q^{-1}\right)} \\ &= \widetilde{[p]}_q z^{p-1} + \sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q a_{j+p} z^{j+p-1}. \qquad (4.4)\end{aligned}$$

Using this representation, we obtain the following relation, which will be essential in what follows.

Proposition 4.1.2. *Given $\tau \geq 0$, the following holds*

$$\tau z \widetilde{D}_q \widetilde{V}^{\beta}_{\tau,q,p}f(z) + (1-\tau\widetilde{[p]}_q)\widetilde{V}^{\beta}_{\tau,q,p}f(z) = \widetilde{V}^{\beta+1}_{\tau,q,p}f(z). \qquad (4.5)$$

Proof.

$$\begin{aligned} & \tau z \widetilde{D}_q \widetilde{V}^{\beta}_{\tau,q,p}f(z) + \left(1-\widetilde{[p]}_q\tau\right)\widetilde{V}^{\beta}_{\tau,q,p}f(z) \\ = \ & \tau z\widetilde{[p]}_q z^{p-1} + \tau z\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q a_{j+p} z^{j+p-1} \\ + \ & \left(1-\widetilde{[p]}_q\tau\right)z^p + \left(1-\widetilde{[p]}_q\tau\right)\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta} a_{j+p} z^{j+p} \\ = \ & z^p + \sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta+1} a_{j+p} z^{j+p} = \widetilde{V}^{\beta+1}_{\tau,q,p}f(z). \end{aligned}$$

So, the equality (4.5) is thus proven. □

Remark 4.1.3. *For $q \to 1^-$, $p = 1$, identity (4.5) becomes the well-known relation associated with the Al-Oboudi differential operator:*

$$z\left(\widetilde{V}^{\beta}_{\tau,1,1}f\right)'(z) = \frac{1}{\tau}\widetilde{V}^{\beta+1}_{\tau,1,1}f(z) - \frac{1-\tau}{\tau}\widetilde{V}^{\beta}_{\tau,1,1}f(z). \qquad (4.6)$$

4.2 Symmetric q-Starlike and Janowski-type Function Classes

The following subclasses of symmetrized q-starlike functions of valency p involving the symmetric q-operator $\widetilde{V}_{\tau,q,p}f(z)$, with respect to functions from the Janowski class, are presented in what follows. From this point onward, we assume that the following condition is satisfied:

$$-1 \leq B < A \leq 1 \qquad (4.7)$$

Definition 4.2.1. *The function $f \in \mathcal{A}_p$ belongs to the class $\widetilde{\mathcal{S}^*}_q^1(\tau, p, \beta, A, B)$ exactly when the next requirement is satisfied:*

$$\operatorname{Re}\left(\frac{(B-1)\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(A-1)}{(B+1)\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(A+1)}\right)\geq 0,$$

that is, by using (4.5)

$$\operatorname{Re}\left(\frac{(B-1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B-1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A-1)}{(B+1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B+1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A+1)}\right)\geq 0.$$

The class $\widetilde{\mathcal{S}^*}_q^1(\tau, p, \beta, A, B)$, will be referred to as Class I of symmetric q-starlike mappings associated with the Janowski family.

Remark 4.2.2. *It is important to highlight that if $q \to 1^-$, $p = 1, \beta = 0$, the class $\widetilde{\mathcal{S}^*}_q^1(\tau, 1, 0, A, B)$ reduces to $\mathcal{S}_q^*(A, B)$*

Now we define the second generalized class.

Definition 4.2.3. *Membership of the function $f \in \mathcal{A}_p$ in the class $\widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B)$ is equivalent to the fulfillment of the following criterion:*

$$\left|\frac{(B-1)\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(A-1)}{(B+1)\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(A+1)}-\frac{1}{1-q^2}\right|<\frac{1}{1-q^2},$$

that is, by using (4.5)

$$\left|\frac{(B-1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B-1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A-1)}{(B+1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B+1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A+1)}-\frac{1}{1-q^2}\right|<\frac{1}{1-q^2}.$$

The class $\widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B)$ will be called Class II of the symmetric q -starlike function family connected with Janowski-type mappings.

Remark 4.2.4. *By taking $q \to 1^-$, $p = 1$, $\beta = 0$, $A = \lambda$ and $B = 0$ in Definition 4.2.3, the resulting class was investigated by Ponnusamy and Singh [15].*

Fnally, we introduce the third generalized class.

Definition 4.2.5. *Membership of $f \in \mathcal{A}_p$ in the class $\widetilde{\mathcal{S}^*}_q^3(\tau, p, \beta, A, B)$ is equivalent to satisfying the condition below:*

$$\left|\frac{(B-1)\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(A-1)}{(B+1)\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(A+1)}-1\right|<1, \tag{4.8}$$

that is, by using (4.5)

$$\left|\frac{(B-1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B-1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A-1)}{(B+1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B+1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A+1)}-1\right|<1. \tag{4.9}$$

We refer to this class, denoted by $\widetilde{\mathcal{S}^*}_q^3(\tau, p, \beta, A, B)$, the Class III of symmetric q-starlike functions corresponding to the Janowski family.

We now turn our attention to a Janowski-type class associated with the operator $\widetilde{V}_{\tau,q,p}^{\beta}$.

Definition 4.2.6. *Let $f \in \mathcal{A}_p$ denote a function from the analytic class $\mathcal{A}_p$. The function f is said to be a member of the class $\widetilde{\mathcal{J}}_q^{\beta}(B,A,\tau,\lambda,p)$ if it fulfills the required subordination condition:*

$$\frac{1}{\widetilde{[p]}_q-\lambda}\left(\frac{z\widetilde{D}_q\widetilde{V}_{\tau,q,p}^{\beta}f(z)}{z^p}-\lambda\right)\prec\frac{1+Bz}{1+Az},\ z\in U, \tag{4.10}$$

where $-1\leq B<A\leq 1$; $0<A\leq 1$ and $0\leq\lambda<\widetilde{[p]}_q$. In view of the standard subordination principle, this is equivalent to a disk condition for the corresponding analytic expression. More precisely, f belongs to the associated class $f\in\widetilde{\mathcal{P}}_q^{\beta}(B,A,\tau,\lambda,p)$ if and only if:

$$\left|\frac{\frac{z\widetilde{D}_q\widetilde{V}_{\tau,q,p}^{\beta}f(z)}{z^p}-\widetilde{[p]}_q}{A\frac{z\widetilde{D}_q\widetilde{V}_{\tau,q,p}^{\beta}f(z)}{z^p}+\lambda(A-B)-\widetilde{[p]}_qB}\right|<1. \tag{4.11}$$

Remark 4.2.7. *In the limiting scenario where $q\to 1^-$, it is observed that by assigning specific values to the parameters β, p, B, and A, the class $\widetilde{\mathcal{J}}_q^{\beta}(B,A,\tau,\lambda,p)$ simplifies to various established subclasses of analytic functions. We list some particular cases below:*

1. The class

$$\widetilde{\mathcal{J}}_1^{0}(B,A,\tau,\lambda,p)=\left\{f\in\mathcal{A}_p:\frac{1}{p-\lambda}\left(\frac{z\left(V_{\tau,p}^{0}f(z)\right)'}{z^p}-\lambda\right)\prec\frac{1+Bz}{1+Az}\right\}$$

was investigated in detail by Aouf (see [1]);
2. The subclasses

$$\widetilde{\mathcal{J}}_1^{0}(-1,1,\tau,\lambda,p)=\left\{f\in\mathcal{A}_p:\frac{1}{p-\lambda}\left(\frac{z\left(V_{\tau,p}^{0}f(z)\right)'}{z^p}-\lambda\right)\prec\frac{1-z}{1+z}\right\}$$

and

$$\widetilde{\mathcal{J}}_1^{0}((2\gamma-1)m,m,\tau,0,p)=\left\{f\in\mathcal{A}_p:\frac{z\left(V_{\tau,p}^{0}f(z)\right)'}{pz^p}\prec\frac{1+(2\gamma-1)mz}{1+mz}\right\}$$

were explored by Owa (see [13, 12]);
3. The class

$$\widetilde{\mathcal{P}}_1^{0}(B,A,\tau,0,p)=\left\{f\in\mathcal{A}_p:\frac{z\left(V_{\tau,p}^{0}f(z)\right)'}{pz^p}\prec\frac{1+Bz}{1+Az}\right\}$$

was analyzed by Chen (see [5]);
4. The class

$$\widetilde{\mathcal{J}}_1^{0}((\gamma-1)n,mn,\tau,0,p)=\left\{f\in\mathcal{A}_p:\frac{z\left(V_{\tau,p}^{0}f(z)\right)'}{pz^p}\prec\frac{1+(\gamma-1)nz}{1+mnz}\right\}$$

was the subject of analysis by Kim and Lee (see [8]);
5. The class

$$\widetilde{\mathcal{J}}_1^{0}((2\gamma-1)m,m,\tau,0,1)=\left\{f\in\mathcal{A}_p:\frac{z\left(V_{\tau,1}^{0}f(z)\right)'}{z}\prec\frac{1+(2\gamma-1)mz}{1+mz}\right\}$$

was considered by Juneja and Mogra (see [7]);
6. The class

$$\widetilde{\mathcal{J}}_1^0\left(B, A, \tau, 0, 1\right) = \left\{ f \in \mathcal{A}_p : \frac{z\left(V_{\tau,1}^0 f(z)\right)'}{z} \prec \frac{1+Bz}{1+Az} \right\}$$

was examined by Mehrok (see [10]);
7. The class

$$\widetilde{\mathcal{J}}_1^0\left(-1, 1, \tau, 0, 1\right) = \left\{ f \in \mathcal{A}_p : \frac{z\left(V_{\tau,1}^0 f(z)\right)'}{z} \prec \frac{1-z}{1+z} \right\}$$

was studied by Mac-Gregor (see [9]);
8. The class

$$\widetilde{\mathcal{J}}_1^0\left(-\gamma, \gamma, \tau, 0, 1\right) = \left\{ f \in \mathcal{A}_p : \frac{z\left(V_{\tau,1}^0 f(z)\right)'}{z} \prec \frac{1-\gamma z}{1+\gamma z} \right\}$$

was investigated by Padmanabhan (see [14]) and by Caplinger and Causey (see [2]).

4.3 Coefficient Inequalities and Basic Inclusion Results

The next result provides a sufficient condition for membership in the class $\widetilde{\mathcal{S}^*}_q^3(\tau, p, \beta, A, B)$ based on a coefficient inequality, which also serves as a sufficient condition for the classes $\widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B)$ and $\widetilde{\mathcal{S}^*}_q^1(\tau, p, \beta, A, B)$.

Theorem 4.3.1. *Let $f \in \mathcal{A}_p$. If the following coefficient inequality is satisfied:*

$$\sum_{j=1}^{\infty} \left(2Y + |B|\right) \left(1 - \tau\widetilde{[p]_q} + \tau\widetilde{[j+p]_q}\right)^{\beta} |a_{j+p}| \leq \left(B+1\right) Z - 2A, \tag{4.12}$$

where the quantities are defined by:

$$A = 1 - \widetilde{[p]_q},\ Y = 1 - \widetilde{[j+p]_q},\ Z = A + 1 - \widetilde{[p]_q},\ B = [j+p]_q\left(B+1\right) - A - 1,$$

then f belongs to the class $\widetilde{\mathcal{S}^*}_q^3(\tau, p, \beta, A, B)$.

Proof. To initiate the proof, we assume that the condition stated in equation (4.12) is satisfied. Under this assumption, it remains to demonstrate that the following implication holds:

$$\left| \frac{\left(B-1\right)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - \left(B-1\right)\left(1 - \tau\widetilde{[p]_q}\right) - \tau\left(A-1\right)}{\left(B+1\right)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - \left(B+1\right)\left(1 - \tau\widetilde{[p]_q}\right) - \tau\left(A+1\right)} - 1 \right| < 1.$$

Thus,

$$\left|\frac{(B-1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B-1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A-1)}{(B+1)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-(B+1)\left(1-\tau\widetilde{[p]}_q\right)-\tau(A+1)}-1\right|$$

$$=\left|\frac{(B-1)\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)-(B-1)\left(1-\tau\widetilde{[p]}_q\right)\widetilde{V}^{\beta}_{\tau,q,p}f(z)-\tau(A-1)\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{(B+1)\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)-(B+1)\left(1-\tau\widetilde{[p]}_q\right)\widetilde{V}^{\beta}_{\tau,q,p}f(z)-\tau(A+1)\widetilde{V}^{\beta}_{\tau,q,p}f(z)}-1\right|$$

$$=2\left|\frac{\left(1+\tau-\tau\widetilde{[p]}_q\right)\widetilde{V}^{\beta}_{\tau,q,p}f(z)-\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{(B+1)\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)-(B+1)\left(1-\tau\widetilde{[p]}_q\right)\widetilde{V}^{\beta}_{\tau,q,p}f(z)-\tau(A+1)\widetilde{V}^{\beta}_{\tau,q,p}f(z)}\right|$$

$$\begin{aligned}
&= 2\left|\frac{\tau\left(1-\widetilde{[p]}_q\right)z^p+\tau\sum_{j=1}^{\infty}\left(1-\widetilde{[j+p]}_q\right)\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}a_{j+p}z^{j+p}}{\tau(B+1)\left(\widetilde{[p]}_q-A-1\right)z^p+\tau\sum_{j=1}^{\infty}\left(A+1-\widetilde{[j+p]}_q(B+1)\right)\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}a_{j+p}z^{j+p}}\right|\\
&\le \frac{2\left\{\tau\left(1-\widetilde{[p]}_q\right)|z^p|+\tau\sum_{j=1}^{\infty}\left(1-\widetilde{[j+p]}_q\right)\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}|a_{j+p}||z^p|\right\}}{\tau(B+1)\left|\widetilde{[p]}_q-A-1\right||z^p|-\tau\sum_{j=1}^{\infty}\left|[j+p]_q(B+1)-A-1\right|\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}|a_{j+p}||z^p|}\\
&\le \frac{2\left\{\tau A+\tau\sum_{j=1}^{\infty}Y\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}|a_{j+p}|\right\}}{\tau(B+1)Z-\tau\sum_{j=1}^{\infty}|B|\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}|a_{j+p}|}\\
&= \frac{2\left\{A+\sum_{j=1}^{\infty}Y\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}|a_{j+p}|\right\}}{(B+1)Z-\sum_{j=1}^{\infty}|B|\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}|a_{j+p}|},
\end{aligned} \tag{4.13}$$

where

$$A=1-\widetilde{[p]}_q,\ Y=1-\widetilde{[j+p]}_q,\ Z=A+1-\widetilde{[p]}_q,\ B=[j+p]_q(B+1)-A-1.$$

he final term in (4.13) does not exceed 1 under the condition (4.12). The desired result is now proven. □

Theorem 4.3.2. *Consider $f\in\mathcal{A}_p$. The function f is a member of the subclass $\widetilde{\mathcal{J}}^{\beta}_q(B,A,\tau,\lambda,p)$ provided that it fulfills the associated condition:*

$$\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\frac{\widetilde{[j+p]}_q}{\widetilde{[p]}_q-\lambda}|a_{j+p}|<\frac{A-B}{1+A}. \tag{4.14}$$

Proof. Under the assumption that (4.14) holds true, it is sufficient to prove that

$$\left|\frac{\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}-\widetilde{[p]}_q}{A\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}+\lambda(A-B)-\widetilde{[p]}_qB}\right|<1.$$

Indeed,

$$\left|\frac{\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}-\widetilde{[p]}_q}{A\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}+\lambda\left(A-B\right)-\widetilde{[p]}_qB}\right|$$

$$=\left|\frac{\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_qa_{j+p}z^{j+p}}{\left(\widetilde{[p]}_q-\lambda\right)\left(A-B\right)z^p+A\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_qa_{j+p}z^{j+p}}\right|$$

$$<\frac{\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q\left|a_{j+p}\right|}{\left(\widetilde{[p]}_q-\lambda\right)\left(A-B\right)-A\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q\left|a_{j+p}\right|}.$$

Provided that condition (4.14) is fulfilled, the preceding inequality is bounded above by 1. As a result, the function f qualifies as a member of the subclass $\widetilde{\mathcal{J}}_q^{\beta}\left(B,A,\tau,\lambda,p\right)$. □

For $A=1$ and $B=-1$, the previous theorems simplifies as follows:

Corollary 4.3.3. *Assume that $f\in\mathcal{A}_p$. The function f lies in the subclass $\widetilde{\mathcal{J}}_q^{\beta}\left(-1,1,\tau,\lambda,p\right)$ whenever the related condition holds:*

$$\sum_{j=1}^{\infty}\left(1-\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\frac{\widetilde{[j+p]}_q}{\widetilde{[p]}_q-\lambda}\left|a_{j+p}\right|<1.$$

Remark 4.3.4. *In connection with Corollary 4.3.3, it is worth emphasizing that the special choice of $B=-1$ and $A=1$ leads to the subordination condition*

$$\frac{1}{\widetilde{[p]}_q-\lambda}\left(\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}-\lambda\right)\prec\frac{1-z}{1+z},\quad z\in U,$$

which corresponds to the Janowski function mapping the unit disk onto the right half-plane. This case highlights a well-known subclass of q-starlike functions associated with the right half-plane, extending and generalizing earlier results obtained in a classical setting. Hence, Corollary 4.3.3 not only simplifies the analytic condition but also establishes a direct bridge between our generalized symmetric q-framework and these classical subclasses.

We now turn to structural relations between the three generalized symmetric q-starlike classes introduced earlier. The following theorem describes a strict chain of inclusions which clarifies the hierarchy among them.

The exposition starts with the inclusion results concerning the classes $\widetilde{\mathcal{S}^*}_q^1\left(\tau,p,\beta,A,B\right)$, $\widetilde{\mathcal{S}^*}_q^2\left(\tau,p,\beta,A,B\right)$, $\widetilde{\mathcal{S}^*}_q^3\left(\tau,p,\beta,A,B\right)$ which represent generalized multivalent symmetric q-starlike function classes associated with Janowski functions.

Theorem 4.3.5. *If the parameters satisfy the condition (4.7), then the following chain of strict class inclusions holds among the extended structure of symmetric q-starlike functions of higher valency related to the Janowski- type functions*

$$\widetilde{\mathcal{S}^*}_q^3\left(\tau,p,\beta,A,B\right)\subset\widetilde{\mathcal{S}^*}_q^2\left(\tau,p,\beta,A,B\right)\subset\widetilde{\mathcal{S}^*}_q^1\left(\tau,p,\beta,A,B\right).\tag{4.15}$$

Proof. Let us consider a function $f \in \widetilde{\mathcal{S}^*}_q^3(\tau, p, \beta, A, B)$. According to Definition 4.2.5, this implies that the condition below is met:

$$\left| \frac{(B-1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B-1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A-1)}{(B+1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B+1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A+1)} - 1 \right| < 1.$$

From this inequality, we can derive the relation:

$$\left| \frac{(B-1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B-1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A-1)}{(B+1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B+1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A+1)} - 1 \right| + \frac{q^2}{1-q^2} < 1 + \frac{q^2}{1-q^2}. \quad (4.16)$$

Now, applying the triangle inequality to the left-hand term of expression (4.16), yields:

$$\left| \frac{(B-1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B-1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A-1)}{(B+1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B+1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A+1)} - \frac{1}{1-q^2} \right| < \frac{1}{1-q^2}. \quad (4.17)$$

Inequality (4.17) satisfies the condition stated in Definition 4.2.3 , which characterizes the class $\widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B)$. Therefore, it follows that

$$f \in \widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B)$$

and consequently, we have the class inclusion:

$$\widetilde{\mathcal{S}^*}_q^3(\tau, p, \beta, A, B) \subset \widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B).$$

Now, let us assume that $f \in \widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B)$. According to Definition 4.2.3, this is equivalent to satisfying the inequality (4.2.3).

Due to the fact that

$$\left| \frac{(B-1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B-1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A-1)}{(B+1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B+1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A+1)} - \frac{1}{1-q^2} \right|$$
$$= \left| \frac{1}{1-q^2} - \frac{(B-1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B-1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A-1)}{(B+1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B+1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A+1)} \right| < \frac{1}{1-q^2},$$

we can express the inequality as:

$$\operatorname{Re}\left(\frac{(B-1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B-1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A-1)}{(B+1)\frac{\widetilde{V}_{\tau,q,p}^{\beta+1} f(z)}{\widetilde{V}_{\tau,q,p}^{\beta} f(z)} - (B+1)\left(1-\tau\widetilde{[p]}_q\right) - \tau(A+1)} \right) \geq 0.$$

This inequality corresponds exactly to the condition required in Definition 4.2.1, for a function to be classified within $\widetilde{\mathcal{S}^*}_q^1(\tau, p, \beta, A, B)$. Hence, we conclude that $f \in \widetilde{\mathcal{S}^*}_q^1(\tau, p, \beta, A, B)$, which establishes the inclusion:

$$\widetilde{\mathcal{S}^*}_q^2(\tau, p, \beta, A, B) \subset \widetilde{\mathcal{S}^*}_q^1(\tau, p, \beta, A, B).$$

Accordingly, the assertion of the theorem has been established. □

The following preliminary lemmas are essential to establish our results for this section.

Lemma 4.3.6 (see [9])**.** *If the constants B_2, B_1, A_1, and A_2 are real and satisfy $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then $\frac{1+A_1 z}{1+B_1 z}$ is subordinate to $\frac{1+A_2 z}{1+B_2 z}$.*

Lemma 4.3.7 (see [16])**.** *Let $a_1 = 1$ and suppose that $a_j \geq 0$ for all $j \geq 2$, with the sequence $\{a_j\}$ being convex and non-increasing; that is, $a_j - 2a_{j+1} + a_{j+2} \geq 0$ and $a_{j+1} + a_{j+2} \geq 0$ for all $j \in \mathbb{N}$. Consequently, the following relation is satisfied throughout the unit disk U:*

$$\operatorname{Re}\left\{\sum_{j=1}^{\infty} a_j z^{j-1}\right\} > \frac{1}{2}.$$

In order to prove the upcoming theorem, we begin by demonstrating the following lemma.

Lemma 4.3.8. *Provided that τ satisfies $\tau \geq \frac{1}{\widetilde{[p]}_q}$, the following inequality holds for all $z \in U$:*

$$\operatorname{Re}\left\{1 + \sum_{j=1}^{\infty} \frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q} z^j\right\} > \frac{1}{2}. \tag{4.18}$$

Proof. Let

$$r(z) = 1 + \sum_{j=1}^{\infty} \frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q} z^j = 1 + \sum_{j=1}^{\infty} Y_j z^j,$$

with

$$Y_j = \frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q},$$

given any $j \geq 1, \tau \geq 0$, $0 < q < 1$, and $p \in \mathbb{N}$. Clearly, $Y_j \geq 0$ for any $j \geq 1$.

A simple calculation shows that

$$Y_{j+1} = \frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p+1]}_q}, \tag{4.19}$$

and

$$Y_{j+2} = \frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p+2]}_q}, \tag{4.20}$$

so

$$Y_{j+1} - Y_{j+2} = \frac{\tau\left(\widetilde{[j+p+2]}_q - \widetilde{[j+p+1]}_q\right)}{\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p+1]}_q\right)\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p+2]}_q\right)} \geq 0. \tag{4.21}$$

By applying (4.19), (4.20), and (4.21), we deduce that

$$\begin{aligned} & Y_j - 2Y_{j+1} + Y_{j+2} \\ = & \frac{\tau X_1 + \tau^2 X_2}{\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p+1]}_q\right)\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p+2]}_q\right)}, \end{aligned}$$

where $X_1 = \left(1 - \tau\widetilde{[p]}_q\right)\left(2\widetilde{[j+p+1]}_q - \widetilde{[j+p]}_q - \widetilde{[j+p+2]}_q\right)$ and
$X_2 = \widetilde{[j+p+1]}_q\widetilde{[j+p+2]}_q - 2\widetilde{[j+p]}_q\widetilde{[j+p+2]}_q + \widetilde{[j+p]}_q\widetilde{[j+p+1]}_q$.

Based on the assumption that $\tau \geq \frac{1}{\widetilde{[p]}_q}$, we conclude that $Y_j - 2Y_{j+1} - Y_{j+2} \geq 0$ for any $j \geq 1$. Consequently, since $\{Y_j\}$ is a monotonically decreasing convex sequence, Lemma 4.3.7 yields

$$\mathrm{Re}\{r(z)\} = \mathrm{Re}\left\{1 + \sum_{j=1}^{\infty} \frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q} z^j\right\} > \frac{1}{2}.$$

Thus, the proof is established. □

Theorem 4.3.9. *If* $\tau \geq \frac{1}{\widetilde{[p]}_q}$,

$$\widetilde{\mathcal{J}}_q^{\beta}(B, A, \tau, \lambda, p) \subseteq \widetilde{\mathcal{J}}_q^{\beta-1}(B, A, \tau, \lambda, p). \tag{4.22}$$

Proof. Assume that $f \in \widetilde{\mathcal{J}}_q^{\beta}(B, A, \tau, \lambda, p)$. Invoking the definition of the class $\widetilde{\mathcal{J}}_q^{\beta}(B, A, \tau, \lambda, p)$, it follows that

$$\frac{1}{\widetilde{[p]}_q - \lambda}\left(\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p} - \lambda\right) \prec \frac{1+Bz}{1+Az}.$$

Through the application of $\widetilde{\mathcal{J}}_q^{\beta-1}(B, A, \tau, \lambda, p)$ and a convolution operation, we arrive at

$$\begin{aligned}
\frac{z\widetilde{D}_q\widetilde{V}^{\beta-1}_{\tau,q,p}f(z)}{\widetilde{[p]}_q z^p} &= \frac{\widetilde{[p]}_q z^p + \sum_{j=1}^{\infty}\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta-1}\widetilde{[j+p]}_q a_{j+p} z^{j+p}}{\widetilde{[p]}_q z^p} \\
&= 1 + \sum_{j=1}^{\infty}\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta-1}\frac{\widetilde{[j+p]}_q}{\widetilde{[p]}_q} a_{j+p} z^j \\
&= \left(1 + \sum_{j=1}^{\infty}\frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q} z^j\right) * \left(1 + \sum_{j=1}^{\infty}\left(1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^{\beta}\frac{\widetilde{[j+p]}_q}{\widetilde{[p]}_q} a_{j+p} z^j\right) \\
&= \left(1 + \sum_{j=1}^{\infty}\frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q} z^j\right) * \frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{[p]}_q z^p}.
\end{aligned}$$

Hence, combining the above expression with (4.10), it follows that

$$\begin{aligned}
&\frac{1}{\widetilde{[p]}_q - \lambda}\left(\frac{z\widetilde{D}_q\widetilde{V}^{\beta-1}_{\tau,q,p}f(z)}{z^p} - \lambda\right) \\
&= \frac{1}{\widetilde{[p]}_q - \lambda}\left\{\widetilde{[p]}_q\left[\left(1 + \sum_{j=1}^{\infty}\frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q} z^j\right) * \frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{\widetilde{[p]}_q z^p}\right] - \lambda\right\} \\
&= \left(1 + \sum_{j=1}^{\infty}\frac{1}{1 - \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q} z^j\right) * \frac{1}{\widetilde{[p]}_q - \lambda}\left(\frac{z\widetilde{D}_q\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p} - \lambda\right) \\
&= r(z) * \frac{1 + Bv(z)}{1 + Av(z)},
\end{aligned}$$

with $v(0) = 0$ and $|v(z)| < 1$. The application of the Herglotz theorem and the previous lemma yields

$$\frac{1}{\widetilde{[p]}_q - \lambda}\left(\frac{z\widetilde{D}_q\widetilde{V}^{\beta-1}_{\tau,q,p}f(z)}{z^p} - \lambda\right) \prec \frac{1+Bz}{1+Az},$$

as $\frac{1+Bz}{1+Az}$ possesses convexity and univalence in U. Accordingly, this implies that

$$\widetilde{\mathcal{J}}^{\beta}_q(B, A, \tau, \lambda, p) \subseteq \widetilde{\mathcal{J}}^{\beta-1}_q(B, A, \tau, \lambda, p).$$

This establishes the claim. □

Theorem 4.3.10. *If the constants B_2, B_1, A_1, and A_2 are real and satisfy $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ and $\beta \geq 0$, it follows that*

$$\widetilde{\mathcal{J}}^{\beta}_q(A_1, B_1, \tau, \lambda, p) \subseteq \widetilde{\mathcal{J}}^{\beta-1}_q(A_2, B_2, \tau, \lambda, p).$$

Proof. Employing Lemma 4.3.6, it follows that

$$\widetilde{\mathcal{J}}^{\beta-1}_q(A_1, B_1, \tau, \lambda, p) \subseteq \widetilde{\mathcal{J}}^{\beta-1}_q(A_2, B_2, \tau, \lambda, p). \tag{4.23}$$

In view of (4.22) and (4.23), it follows that

$$\widetilde{\mathcal{J}}^{\beta}_q(A_1, B_1, \tau, \lambda, p) \subseteq \widetilde{\mathcal{J}}^{\beta-1}_q(A_1, B_1, \tau, \lambda, p) \subseteq \widetilde{\mathcal{J}}^{\beta-1}_q(A_2, B_2, \tau, \lambda, p).$$

Hence, we get

$$\widetilde{\mathcal{J}}^{\beta}_q(A_1, B_1, \tau, \lambda, p) \subseteq \widetilde{\mathcal{J}}^{\beta-1}_q(A_2, B_2, \tau, \lambda, p).$$

Therefore, the desired result is proved. □

4.4 Second Order Differential Subordination and Superordination for the Symmetric q-Operator $\widetilde{V}^{\beta}_{\alpha,q,p}$

The lemmas below, which are symmetric q-analogues of known results for the classical derivative, will be instrumental in deriving second-order differential subordination results.

Lemma 4.4.1. *Consider a one-to-one analytic function u defined in U and let θ and ϕ be holomorphic functions on a domain D that contains the image $u(U)$ with the condition that $\phi(w) \neq 0$, for all $w \in u(U)$. Define $Q(z) = z\widetilde{D}_q u(z)\phi(u(z))$ and $h(z) = \theta(u(z)) + Q(z)$. Assume the following conditions hold:*

1. *Q is a one-to-one starlike function within U and*
2. *$\mathrm{Re}\left(\frac{z\widetilde{D}_q h(z)}{Q(z)}\right) > 0$ for all $z \in U$.*

If s is analytic in U, satisfies $s(0) = u(0)$, $s(U) \subseteq D$ and

$$\theta(s(z)) + z\widetilde{D}_q s(z)\phi(s(z)) \prec \theta(u(z)) + z\widetilde{D}_q u(z)\phi(u(z)) = h(z), \tag{4.24}$$

then $s(z) \prec u(z)$ and $u(z)$ represents the optimal dominant.

Proof. Suppose u is analytic and one-to-one in U, and define $Q(z) = z\widetilde{D}_q u(z)\phi(u(z))$, $h(z) = \theta(u(z)) + Q(z)$, where θ and ϕ are holomorphic functions in a region D that includes $u(U)$ such that $\phi(w)$ non-vanishing on $u(U)$, for all $w \in u(U)$. Assume that conditions 1 and 2 hold. Define $m(z) = \theta(s(z)) + z\widetilde{D}_q s(z)\phi(s(z))$, where s is holomorphic in U, $s(0) = u(0)$, $s(U) \subseteq D$. By hypotheses $m(z) \prec h(z)$.

When q approaches 1^-, the symmetric q-operator $\widetilde{D}_q f(z)$ reduces to the classical derivative $f'(z)$. Therefore, (4.24) becomes:

$$\theta(s(z)) + zs'(z)\,\phi(s(z)) \prec \theta(u(z)) + zu'(z)\,\phi(u(z)).$$

Applying the known lemma for the classical derivative, it can be concluded that $s(z) \prec u(z)$ and u represent the principal dominant. □

We are now in a position to state the first main result of this section, which concerns a second-order differential subordination involving the symmetric q-differential operator $\widetilde{V}^{\beta}_{\tau,q,p}$.The first main result presented in this section concerns a second-order differential subordination involving the symmetric q-differential operator $\widetilde{V}^{\beta}_{\tau,q,p}$. This theorem establishes sufficient conditions under which a function transformed by this operator is subordinate to a given univalent function. A significant feature of this result is that it not only guarantees the subordination relation but also explicitly identifies the best dominant — that is, the minimal univalent function (in the subordination sense) to which all admissible solutions are subordinate. This sharp bound enhances the applicability of the theorem and reflects the precision afforded by the use of the symmetric q-calculus framework.

Theorem 4.4.2. *Let $f \in \mathcal{A}_p$, and suppose that the function $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}$ belongs to $\mathcal{H}(U)$, where $z \in U$. Let $u(z)$ be a convex univalent function in the open unit disk, normalized, satisfying $u(0) = 1$. Consider that for some constants $a, b, c, \in \mathbb{C}$, with $c \neq 0$, the following condition is satisfied:*

$$\operatorname{Re}\left\{\frac{a}{c} + \frac{b}{c}\left[u(qz) + u\left(q^{-1}z\right)\right] + \frac{\widetilde{D}_q u\left(q^{-1}z\right)}{\widetilde{D}_q u(z)} + \frac{qz\widetilde{D}^{(2)}_q u(z)}{\widetilde{D}_q u(z)}\right\} > 0, \quad z \in U. \tag{4.25}$$

Let us describe the function $\Psi^{\beta}_{\tau,q,p}(a, b, c; z)$ as

$$\begin{aligned}\Psi^{\beta}_{\tau,q,p}(a, b, c; z) &= \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}\left(a + b\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}\right) \\ &\quad + \frac{c}{\tau}\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(qz)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)}\left[\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z)} - \left(1 - \tau\widetilde{[p]_q}\right)\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z)}\right] \\ &\quad - \frac{c}{\tau}\left[\frac{\widetilde{V}^{\beta+2}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)} - \left(1 - \tau\widetilde{[p]_q}\right)\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)}\right].\end{aligned} \tag{4.26}$$

If the subordination

$$\Psi^{\beta}_{\tau,q,p}(a, b, c; z) \prec au(z) + b\,(u(z))^2 + cz\widetilde{D}_q u(z) \tag{4.27}$$

holds in U, then the following subordination result is valid::

$$\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} \prec u(z), \tag{4.28}$$

and u is the best dominant.

Proof. Let

$$s(z) := \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}, \tag{4.29}$$

where $z \in U \backslash \{0\}$, $f \in \mathcal{A}_p$. Clearly, s is analytic U and $s(0) = 1$. By applying the symmetric q-differentiating to $s(z)$, we find

$$z\widetilde{D}_q s(z) = \frac{z\widetilde{D}_q \widetilde{V}^{\beta}_{\tau,q,p} f(z) \widetilde{V}^{\beta+1}_{\tau,q,p} f(qz) - z\widetilde{D}_q \widetilde{V}^{\beta+1}_{\tau,q,p} f(z) \widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z) \widetilde{V}^{\beta}_{\tau,q,p} f(qz)}.$$

Invoking identity (4.5), this expression simplifies to:

$$z\widetilde{D}_q s(z) = \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(qz)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)} \left(\frac{1}{\tau} \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z)} - \frac{1-\tau\widetilde{[p]}_q}{\tau} \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z)} \right)$$

$$-\frac{1}{\tau} \frac{\widetilde{V}^{\beta+2}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)} + \frac{1-\tau\widetilde{[p]}_q}{\tau} \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)}. \tag{4.30}$$

Now define the auxiliary functions: $\theta(w) = aw + bw^2$ and $\phi(w) = c$, where $a, b, c \in \mathbb{C}$ and $c \neq 0$. It is clear that θ is holomorphic on $\mathbb{C} \backslash \{0\}$, ϕ is likewise holomorphic and non-vanishing on the same domain. Set $Q(z) = z\widetilde{D}_q u(z) \phi(u(z)) = cz\widetilde{D}_q u(z)$ and $h(z) = \theta(u(z)) + Q(z) = au(z) + b(u(z))^2 + cz\widetilde{D}_q u(z)$, $z \in U$. By direct computation, we obtain:
$\operatorname{Re}\left(\frac{z\widetilde{D}_q h(z)}{Q(z)} \right) = \operatorname{Re}\left\{ \frac{a}{c} + \frac{b}{c}\left[u(qz) + u(q^{-1}z)\right] + \frac{\widetilde{D}_q u(q^{-1}z)}{\widetilde{D}_q u(z)} + \frac{qz\widetilde{D}^{(2)}_q u(z)}{\widetilde{D}_q u(z)} \right\} > 0$. Applying (4.30), we have

$$as(z) + b(s(z))^2 + cz\widetilde{D}_q s(z) = \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} \left(a + b\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} \right)$$

$$+ \quad \frac{c}{\tau} \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(qz)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)} \left[\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z)} - \left(1 - \tau\widetilde{[p]}_q\right) \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(q^{-1}z)} \right]$$

$$- \quad \frac{c}{\tau} \left[\frac{\widetilde{V}^{\beta+2}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)} - \left(1 - \tau\widetilde{[p]}_q\right) \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(qz)} \right].$$

Based on (4.27), we conclude that

$$as(z) + b(s(z))^2 + cz\widetilde{D}_q s(z) \prec au(z) + b(u(z))^2 + cz\widetilde{D}_q u(z).$$

Hence, the conditions of the subordination from Lemma 4.4.1 are satisfied and it follows that $s(z) \prec u(z)$, for all $z \in U$; that is, $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} \prec u(z)$, $z \in U$, with u being the best dominant. □

Corollary 4.4.3. *Let $u(z) = \frac{1+Az}{1+Bz}$, and suppose that conditions (4.7) and (4.25) are satisfied. If $f \in \mathcal{A}_p$ and*

$$\Psi^{\beta}_{\tau,q,p}(a,b,c;z) \prec a\frac{1+Az}{1+Bz} + b\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{cz\left[ABz\left(q^2-1\right) - (A-B)q\right]}{q + Bz\left(q^2+1\right) + qB^2z^2},$$

for $a, b, c \in \mathbb{C}$, $c \neq 0$, under the condition that $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ is formulated on (4.26), then it follows that

$$\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in U,$$

and $\frac{1+Az}{1+Bz}$ represents the optimal dominant.

Proof. Substituting $u(z) = \frac{1+Az}{1+Bz}$ into Theorem 4.4.2 yields the stated corollary as a direct consequence. □

Remark 4.4.4. *The function* $u(z) = \frac{1+Az}{1+Bz}$ *defines a conformal mapping of* U *into a circular area in the complex plane exhibiting symmetry relative to the real axis. Specifically, the region covered by* U *under* $u(z) = \frac{1+Az}{1+Bz}$ *is a disk with center at* $\frac{1-AB}{1-B^2}$ *and radius* $\frac{A-B}{1-B^2}$*. In the special case where* $B = -1$*, the image becomes a half-plane. The endpoints of the diameter of the image disc on the real axis are given by:* $M = \left(\frac{1-A}{1-B}, 0\right)$ *and* $N = \left(\frac{1+A}{1+B}, 0\right)$ *. These mappings, often referred to as Janowski functions, are widely used in geometric complex analysis, particularly in a focused inquiry into subfamilies of analytic functions with univalence and starlikeness.*

Corollary 4.4.5. *Assume that* $u(z) = \frac{1+(1-2\gamma)z}{1-z}$, $0 \leq \gamma < 1$ *and that condition (4.25) is satisfied. If* $f \in \mathcal{A}_p$ *and*

$$\Psi^{\beta}_{\tau,q,p}(a,b,c;z) \prec a\frac{1+(1-2\gamma)z}{1-z}+b\left(\frac{1+(1-2\gamma)z}{1-z}\right)^2+\frac{cz\left[z(1-2\gamma)(1-q^2)-2(1-\gamma)q\right]}{q-z(q^2+1)+qz^2},$$

given $a, b, c \in \mathbb{C}$, $c \neq 0$*, and assuming that* $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ *is specified in (4.26), then we conclude that*

$$\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)} \prec \frac{1+(1-2\gamma)z}{1-z},$$

and $\frac{1+(1-2\gamma)z}{1-z}$ *represents the optimal dominant.*

Proof. Letting $u(z) = \frac{1+(1-2\gamma)z}{1-z}$, with $0 \leq \gamma < 1$ in Theorem 4.4.2, the conclusion of this corollary readily follows as a particular case. □

We now illustrate Theorem 5.5.0.2. by considering several important choices of the dominant function u, all of Janowski type.

Corollary 4.4.6. *Let* $u(z) = \frac{1+z}{1-z}$ *and suppose that condition (4.25) is satisfied. If* $f \in \mathcal{A}_p$ *and the subordination*

$$\Psi^{\beta}_{\tau,q,p}(a,b,c;z) \prec a\frac{1+z}{1-z}+b\left(\frac{1+z}{1-z}\right)^2+\frac{cz\left[z(1-q^2)-2q\right]}{q-z(q^2+1)+qz^2}$$

holds for $a, b, c \in \mathbb{C}$*, with* $c \neq 0$*, where* $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ *is defined in (4.26), it can be concluded that*

$$\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

Proof. This result arises as a straightforward implication of Theorem 4.4.2 by selecting the function $u(z) = \frac{1+z}{1-z}$, which serves as a specific instance satisfying the conditions of the theorem. □

Corollary 4.4.7. *Let* $u(z) = \frac{1+z}{(1-z)^2}$, $z \in U$ *and suppose that condition (4.25) is satisfied. If* $f \in \mathcal{A}_p$ *and the subordination*

$$\Psi^{\beta}_{\tau,q,p}(a,b,c;z) \prec a\frac{1+z}{(1-z)^2}+b\left(\frac{1+z}{(1-z)^2}\right)^2+\frac{cz\left[(1+qz)\left(\widetilde{[2]}_q z-2\right)-(1-q^{-1}z)^2\right]}{\left(1-\widetilde{[2]}_q z+z^2\right)^2}$$

holds for $a, b, c \in \mathbb{C}$, with $c \neq 0$, where $\Psi^{\beta}_{\tau,q,p}(a, b, c; z)$ is defined in (4.25), then it follows that

$$\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} \prec \frac{1+z}{(1-z)^2},$$

and $\frac{1+z}{(1-z)^2}$ represents the optimal dominant.

Proof. This result is derived by evaluating Theorem 4.4.2 at the specific function $u(z) = \frac{1+z}{(1-z)^2}$, which leads directly to the stated conclusion. □

We next turn to a related but technically simpler subordination problem involving the ratio $\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$.

Theorem 4.4.8. *Assume that $f \in \mathcal{A}_p$ and that the function $\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$ belongs to the class $\mathcal{H}(U)$, where $z \in U\backslash\{0\}$. Suppose that $u(z)$ is a convex univalent function in U, normalized by the condition $u(0) = 1$. Let $m, n \in \mathbb{C}$, with $n \neq 0$, and suppose that the following condition is satisfied:*

$$\operatorname{Re}\left\{\frac{m}{n} + n + \frac{z\widetilde{D}^{(2)}_q u(z)}{\widetilde{D}_q u(z)}\right\} > 0, \quad z \in U. \tag{4.31}$$

The function $\psi^{\beta}_{\tau,q,p}(m, n; z)$ is given by

$$\psi^{\beta}_{\tau,q,p}(m, n; z) = \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}\left[m + \frac{n}{\tau}\left(\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} - 1\right)\right]. \tag{4.32}$$

If the subordination

$$\psi^{\beta}_{\tau,q,p}(m, n; z) \prec mu(z) + nz\widetilde{D}_q u(z), \tag{4.33}$$

is satisfied in U, then the subordination relation

$$\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p} \prec u(z), \quad z \in U\backslash\{0\}, \tag{4.34}$$

holds and u represents the finest majorant.

Proof. Let the following auxiliary expression be given:

$$s(z) = \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}, \tag{4.35}$$

where $z \in U\backslash\{0\}$, $f \in \mathcal{A}_p$. It is evident that z is analytic in U and satisfies the normalization criterion $s(0) = 1$.

By applying logarithmic symmetric q-differentiation to $s(z)$, on both sides of (4.35), we obtain

$$\frac{z\widetilde{D}_q s(z)}{s(z)} = \frac{z\widetilde{D}_q \widetilde{V}^{\beta}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} - \widetilde{[p]}_q.$$

Using identity (4.5), this expression reduces to

$$\frac{z\widetilde{D}_q s(z)}{s(z)} = \frac{1}{\tau}\left(\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} - 1\right). \tag{4.36}$$

We now introduce two auxiliary functions defined by $\theta(w) := aw$ and $\phi(w) := b$, where $m, n \in \mathbb{C}$ and $n \neq 0$. The function θ is clearly analytic in $\mathbb{C}\backslash\{0\}$, and ϕ, being constant and nonzero, is also analytic in $\mathbb{C}\backslash\{0\}$ and non-vanishing in $\mathbb{C}\backslash\{0\}$.

Next, we define the function $Q(z) = z\widetilde{D}_q u(z)\,\phi(u(z)) = nz\widetilde{D}_q u(z)$, and consequently, we define $h(z) = \theta(u(z)) + Q(z) = mu(z) + nz\widetilde{D}_q u(z)$.

Through straightforward calculation, we arrive at the following result:

$$\operatorname{Re}\left(\frac{z\widetilde{D}_q h(z)}{Q(z)}\right) =$$

$$\operatorname{Re}\left\{\frac{m}{n} + b + \frac{z\widetilde{D}_q^{(2)} u(z)}{\widetilde{D}_q u(z)}\right\} > 0.$$

By substituting from identity (4.36), we obtain the following relation:

$$ms(z) + nz\widetilde{D}_q s(z) = m\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p} + \frac{n}{\tau}\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}\left(\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} - 1\right)$$

$$= \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}\left[m + \frac{n}{\tau}\left(\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} - 1\right)\right].$$

Based on (4.33), we can conclude that

$$ms(z) + nz\widetilde{D}_q s(z) \prec mu(z) + nz\widetilde{D}_q u(z).$$

Therefore, the requirements of the subordination in Lemma 4.4.1 are fully met, which implies that $s(z) \prec u(z)$ for all $z \in U\backslash\{0\}$. In other words, $\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p} \prec u(z)$, $z \in U\backslash\{0\}$, where u serves as the best dominant for this subordination. □

Corollary 4.4.9. *Let $u(z) = \frac{1+Az}{1+Bz}$ and $-1 \leq B < A \leq 1$ and suppose that condition (4.31) is satisfied. If $f \in \mathcal{A}_p$ and*

$$\psi^{\beta}_{\tau,q,p}(m, n; z) \prec m\frac{1+Az}{1+Bz} + nz\frac{A-B}{(1+Bqz)(1+Bq^{-1}z)},$$

for the complex constants $m, n \in \mathbb{C}$ and $n \neq 0$, $-1 \leq B < A \leq 1$, where the function $\psi^{\beta}_{\tau,q,p}(m, n; z)$ is defined in (4.32), then the following subordination holds:

$$\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \quad z \in U,$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. By choosing $u(z) = \frac{1+Az}{1+Bz}$, with parameters in the range of $-1 \leq B < A \leq 1$, and applying Theorem 4.4.8, the desired conclusion follows immediately as a particular case. □

Remark 4.4.10. *The function $u(z) = \frac{1+Az}{1+Bz}$, $z \in U$, where $-1 \leq B < A \leq 1$, was originally introduced by Janowski [6]. It defines a conformal mapping of U onto a circular area in $\mathbb{C}$, exhibiting reflectional symmetry across the real line. Specifically, the image of U under $u(z) = \frac{1+Az}{1+Bz}$ is a disk centered at$\frac{1-AB}{1-B^2}$ with a radius of $\frac{A-B}{1-B^2}$. In the limiting case when $B = -1$, the image degenerates into a right half-plane. The real axis intercepts of the circular image– i.e., the endpoints of the diameter — are located at $M = \left(\frac{1-A}{1-B}, 0\right)$ and $N = \left(\frac{1+A}{1+B}, 0\right)$. These mappings, commonly referred to as Janowski functions, play a fundamental role in geometric function theory. They are extensively used in the characterization and analysis of various subclasses of univalent, starlike, and convex functions.*

Corollary 4.4.11. *Suppose that* $u(z) = \frac{1+(1-2\gamma)z}{1-z}$, $0 \leq \gamma < 1$ *and that condition (4.31) is satisfied. If* $f \in \mathcal{A}_p$ *and the function* $\psi^{\beta}_{\tau,q,p}(m,n;z)$, *as given in (4.32), satisfies the subordination relation*

$$\psi^{\beta}_{\tau,q,p}(m,n;z) \prec m\frac{1+(1-2\gamma)z}{1-z} + nz\frac{2(1-\gamma)}{(1-qz)(1-q^{-1}z)},$$

for some complex constants, $m, n \in \mathbb{C}$, *and* $n \neq 0$, *then the subordination*

$$\frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p} \prec \frac{1+(1-2\gamma)z}{1-z},$$

holds, and the function $\frac{1+(1-2\gamma)z}{1-z}$ *is the best dominant.*

Proof. When choosing $u(z) = \frac{1+(1-2\gamma)z}{1-z}$, with $0 \leq \gamma < 1$ in Theorem 4.4.8, the stated result follows immediately as a particular instance of the general theorem. □

Corollary 4.4.12. *Let* $u(z) = \frac{1+z}{1-z}$ *and assume that condition (4.31) is satisfied. Suppose that* $f \in \mathcal{A}_p$ *and the function* $\psi^{\beta}_{\tau,q,p}(m,n;z)$, *defined in (4.32), satisfies the subordination*

$$\psi^{\beta}_{\tau,q,p}(m,n;z) \prec m\frac{1+z}{1-z} + nz\frac{2}{(1-qz)(1-q^{-1}z)},$$

for some complex constants, $m, n \in \mathbb{C}$, *with* $n \neq 0$. *Then it follows that*

$$\frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ *represents the best dominant.*

Proof. This conclusion is an immediate consequence of Theorem 4.4.8 when selecting the specific function $u(z) = \frac{1+z}{1-z}$, which satisfies the hypotheses of the theorem and corresponds to a particular case within its framework. □

Corollary 4.4.13. *Let* $u(z) = \frac{1+z}{(1-z)^2}$, $z \in U$ *and suppose that condition (4.31) is satisfied. If* $f \in \mathcal{A}_p$ *and the function* $\psi^{\beta}_{\tau,q,p}(m,n;z)$, *as defined in (4.31), satisfies the subordination*

$$\psi^{\beta}_{\tau,q,p}(m,n;z) \prec m\frac{1+z}{(1-z)^2} + nz\frac{(q^{-1}-q)z^2 + (q^{-2}-q^2)z - 3(q^{-1}-q)}{(q-q^{-1})(1-qz)^2(1-q^{-1}z)^2},$$

for $m, n, c \in \mathbb{C}$, *with* $c \neq 0$, *then the subordination relation*

$$\frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p} \prec \frac{1+z}{(1-z)^2},$$

holds, and the function $\frac{1+z}{(1-z)^2}$ *serves as the best dominant.*

Proof. The result is obtained as a direct consequence of Theorem 4.4.8 by choosing $u(z) = \frac{1+z}{(1-z)^2}$, which satisfies the theorem's assumptions and yields the stated conclusion as a particular case. □

Next, we consider the dual problem to differential subordination, namely superordination, and then combine both directions to derive sandwich-type results. The basic tool is the following lemma, which is a symmetric q-analogue of a classical result for first-order differential superordination.

Lemma 4.4.14. *Suppose u is convex and injective in U and v and ϕ are holomorphic within a domain D that contains the image $u(U)$. Let us assume:*

1. $\operatorname{Re}\left(\frac{\widetilde{D}_q v(u(z))}{\phi(u(z))}\right) > 0$*, given any $z \in U$, and*

2. $\Psi(z) = z\widetilde{D}_q u(z)\phi(u(z))$ *is analytic, injective, and starlike in the open unit disk. Assume that $s(z) \in \mathcal{H}[u(0), 1] \cap Q$, where $s(U) \subseteq D$ and the composed mapping $v(s(z)) + z\widetilde{D}_q s(z)\phi(s(z))$ is injective in the unit disk and*

$$v(u(z)) + z\widetilde{D}_q u(z)\phi(u(z)) \prec v(s(z)) + z\widetilde{D}_q s(z)\phi(s(z)),$$

then $u(z) \prec s(z)$ and u is identified as the extremal solution among all subordinants.

Proof. This proof mirrors the method used in the previous lemma. □

The subsequent theorem addresses a second-order differential superordination involving the symmetric q-differential operator $\widetilde{V}^{\beta}_{\tau,q,p}$. It provides sufficient conditions under which a given univalent function is superordinate to a transformed analytic function. A key contribution of this result lies in the identification of the best subordinant — that is, the largest function (with respect to subordination) among all admissible subordinants satisfying the differential inequality. This result not only complements the earlier subordination theorem but also highlights the duality structure inherent in the theory. The use of symmetric q-calculus facilitates a more refined formulation of the problem, allowing for greater generality and improved analytical control over the behavior of the solutions.

Theorem 4.4.15. *Let u be an analytic and injective mapping on the open unit disk U, with the property of convexity, satisfying $u(0) = 1$. Suppose that for constants $a, b, c \in \mathbb{C}$, where $c \neq 0$, the following condition holds:*

$$\operatorname{Re}\left(\frac{a}{c} + \frac{b}{c}\left[u(qz) + u\left(q^{-1}z\right)\right]\right) > 0. \tag{4.37}$$

Assume that $f \in \mathcal{A}_p$ and that the function $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}$ belongs to the class $\mathcal{H}[u(0), 1] \cap Q$. Suppose also that the function $\Psi^{\beta}_{\tau,q,p}(a, b, c; z)$, defined in (4.26), is univalent in U. If the subordination

$$au(z) + b(u(z))^2 + cz\widetilde{D}_q u(z) \prec \Psi^{\beta}_{\tau,q,p}(a, b, c; z), \tag{4.38}$$

is satisfied, then it follows that

$$u(z) \prec \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}, \tag{4.39}$$

and u is the best subordinant.

Proof. Let $s(z) = \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}$, $z \in U\backslash\{0\}$, $f \in \mathcal{A}_p$. Define the auxiliary functions: $v(v) = av + bv^2$ and $\phi(v) = c$. Clearly, v is holomorphic in $\mathbb{C}$ and ϕ is analytic except at the origin and non-vanishing in $\mathbb{C}\backslash\{0\}$. A straightforward computation shows that:

$$\frac{\widetilde{D}_q v(u(z))}{\phi(u(z))} = \frac{a}{c} + \frac{b}{c}\left[u(qz) + u\left(q^{-1}z\right)\right].$$

By hypothesis (15), the real part of this expression is positive for all $z \in U$, that is:

$$\operatorname{Re}\left(\frac{\widetilde{D}_q v(u(z))}{\phi(u(z))}\right) = \operatorname{Re}\left(\frac{a}{c} + \frac{b}{c}\left[u(qz) + u\left(q^{-1}z\right)\right]\right) > 0$$

Using the subordination assumption (4.38), we obtain

$$au(z) + b(u(z))^2 + cz\widetilde{D}_q u(z) \prec as(z) + b(s(z))^2 + cz\widetilde{D}_q s(z).$$

Now, applying Lemma 4.4.14, it follows that:

$$u(z) \prec s(z) = \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}, \quad z \in U$$

and that u is the best subordinant. □

Several corollaries can be obtained again by choosing Janowski-type functions for u. In each case, the resulting statement asserts that the given Janowski mapping is the best subordinant for the corresponding family of analytic functions transformed by $\widetilde{V}^{\beta}_{\tau,q,p}$.

Corollary 4.4.16. *Let $u(z) = \frac{1+Az}{1+Bz}$, where $-1 \le B < A \le 1$. Assume that condition (4.37) is satisfied. If $f \in \mathcal{A}_p$ and the function $s(z) = \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}$ is included in the class $\mathcal{H}[u(0), 1] \cap Q$, and if the subordination*

$$a\frac{1+Az}{1+Bz} + b\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{cz\left[ABz\left(q^2-1\right) - (A-B)q\right]}{q + Bz\left(q^2+1\right) + qB^2z^2} \prec \Psi^{\beta}_{\tau,q,p}(a,b,c;z)$$

holds in U, for constants $a, b, c \in \mathbb{C}$, with $c \neq 0$, where $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ is defined in (4.26), then the following subordination relation holds:

$$\frac{1+Az}{1+Bz} \prec \frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}, \; z \in U,$$

and $u(z) = \frac{1+Az}{1+Bz}$ represents the optimal subordinant.

Proof. The assertion is a direct consequence of Theorem 4.4.15, with the choice $u(z) = \frac{1+Az}{1+Bz}$, under the stated assumptions. □

Corollary 4.4.17. *Assume that $u(z) = \frac{1+(1-2\gamma)z}{1-z}, 0 \le \gamma < 1, z \in U$ and that condition (4.37) is satisfied. If $f \in \mathcal{A}_p$ and $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)} \in \mathcal{H}[u(0), 1] \cap Q$ and*

$$a\frac{1+(1-2\gamma)z}{1-z} + b\left(\frac{1+(1-2\gamma)z}{1-z}\right)^2 + \frac{cz\left[z(1-2\gamma)\left(1-q^2\right) - 2(1-\gamma)q\right]}{q - z\left(q^2+1\right) + qz^2} \prec \Psi^{\beta}_{\tau,q,p}(a,b,c;z),$$

for $a, b, c \in \mathbb{C}$, $c \neq 0$,, $0 < \gamma \le 1$, where $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ is formulated (4.26), then we conclude that

$$\frac{1+(1-2\gamma)z}{1-z} \prec \frac{\widetilde{V}^{(\beta+1)}_{\tau,q,p} f(z)}{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}$$

and $\frac{1+(1-2\gamma)z}{1-z}$ represents the optimal subordinant.

Proof. Corollary follows through the use of Theorem 4.4.15 for $u(z) = \frac{1+(1-2\gamma)z}{1-z}$, $0 < \gamma \le 1$. □

Corollary 4.4.18. *Let* $u(z)=\frac{1+z}{1-z}$, $z\in U$ *and suppose that condition (4.37) is satisfied. If* $f\in\mathcal{A}_p$, *and define the function* $s(z)=\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}$. *Suppose further that* $s(z)\in\mathcal{H}[u(0),1]\cap Q$ *and that the subordination*

$$a\frac{1+z}{1-z}+b\left(\frac{1+z}{1-z}\right)^2+\frac{cz\left[z\left(1-q^2\right)-2q\right]}{q-z\left(q^2+1\right)+qz^2}\prec\Psi^{\beta}_{\tau,q,p}(a,b,c;z),$$

holds for some complex constants a,b,c, *where* $c\neq0$, *and assuming that* $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ *is introduced as in (4.26), the subsequent subordination statement holds:*

$$\frac{1+z}{1-z}\prec\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)},$$

and $\frac{1+z}{1-z}$ *represents the sharpest subordinant.*

Proof. The result is obtained as a direct consequence of Theorem 4.4.15, by choosing $u(z)=\frac{1+z}{1-z}$, which is a conformal convex mapping on U, so that $u(0)=1$. The assumptions ensure that all hypotheses of the theorem are fulfilled, hence the conclusion follows. □

Corollary 4.4.19. *Let* $u(z)=\frac{1+z}{(1-z)^2}$, $z\in U$ *and suppose that condition (4.37) is satisfied. If* $f\in\mathcal{A}_p$ *and define the function* $s(z)=\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}$. *Suppose further that* $s(z)\in\mathcal{H}[u(0),1]\cap Q$ *and that the subordination*

$$a\frac{1+z}{(1-z)^2}+b\left(\frac{1+z}{(1-z)^2}\right)^2+\frac{cz\left[(1+qz)\left(\widetilde{[2]}_qz-2\right)-\left(1-q^{-1}z\right)^2\right]}{\left(1-\widetilde{[2]}_qz+z^2\right)^2}\prec\Psi^{\beta}_{\tau,q,p}(a,b,c;z),$$

holds for $a,b,c\in\mathbb{C}$, *with* $c\neq0$, *where* $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ *is defined in (4.26). Accordingly, the subsequent subordination statement holds true::*

$$\frac{1+z}{(1-z)^2}\prec\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)},$$

and $\frac{1+z}{(1-z)^2}$ *constitutes the sharpest subordinant.*

Proof. This outcome follows directly from Theorem 4.4.15, by selecting $u(z)=\frac{1+z}{(1-z)^2}$, a one-to-one convex holomorphic mapping defined in U, satisfying $u(0)=1$. Under the given assumptions, all the premises of the theorem are fulfilled, and the outcome is thereby established. □

By simultaneously invoking the conclusions of Theorem 4.4.2 and Theorem 4.4.15, we derive a comprehensive result in the form of a sandwich-type theorem. This theorem encapsulates both the subordination and superordination frameworks, yielding a two-sided inclusion for the analytic function under consideration. Specifically, it establishes that the function transformed by the symmetric q-differential operator lies between two extremal functions — designated as the optimal subordinant and dominant, respectively — each characterized by precise geometric or analytic conditions. The result provides a sharp and symmetric containment of the target function, demonstrating the power of combining differential subordination and superordination theories within the structure of symmetric q-calculus.

The general formulation and its Janowski specializations are given in the theorem and corollaries that follow in your text (for u_1 and u_1 of the form $\frac{1+Az}{1+Bz}$ or $\frac{1+(1-2\gamma)z}{1-z}$), and we do not repeat the detailed formulas here, as they remain unchanged.

Theorem 4.4.20. *Consider u_1 and u_2 as injective analytic mappings in the unit disk U, such that $u_1(z) \neq 0$ and $u_2(z) \neq 0$, for each $z \in U$. Assume further that the functions $z\widetilde{D}_q u_1(z)$ and $z\widetilde{D}_q u_1(z)$ are starlike and univalent in U. Suppose that the functions u_1 satisfies (4.25) and u_2 satisfies (4.37) , $f \in \mathcal{A}_p$, and the function $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}$ is included in the class $\mathcal{H}[u(0),1] \cap Q$. Assume also that the function $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ defined in Theorem 4.4.15 maps U conformally onto its image, and that for constants $a,b,c \in \mathbb{C}$, with $c \neq 0$, the following double subordination holds:*

$$au_1(z) + b(u_1(z))^2 + cz\widetilde{D}_q u_1(z) \prec \Psi^{\beta}_{\tau,q,p}(a,b,c;z) \prec au_2(z) + b(u_2(z))^2 + cz\widetilde{D}_q u_2(z),$$

Then it follows that:

$$u_1(z) \prec \frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)} \prec u_2(z),$$

and the functions u_1 and u_2 serve, respectively, as the sharp subordinant and dominant associated with this sandwich-type relation.

Let $u_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $u_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where the parameters satisfy the ordering $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$. Under these conditions, the following corollary holds.

Corollary 4.4.21. *Assume that the conditions specified in (4.25) and (4.37) hold for the functions $u_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $u_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively, where the parameters satisfy $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Let $f \in \mathcal{A}_p$, and suppose the function $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}$ belongs to the class $\mathcal{H}[u(0),1] \cap Q$. Assume further that the function $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ defined in (4.26), is univalent in U, and that for constants $a,b,c \in \mathbb{C}$, with $c \neq 0$, the following double subordination holds:*

$$a\frac{1+A_1 z}{1+B_1 z} + b\left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \frac{cz\left[A_1 B_1 z\left(q^2-1\right) - \left(A_1 - B_1\right)q\right]}{q + B_1 z\left(q^2+1\right) + qB_1^2 z^2} \prec \Psi^{\beta}_{\tau,q,p}(a,b,c;z)$$

$$\prec a\frac{1+A_2 z}{1+B_2 z} + b\left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \frac{cz\left[A_2 B_2 z\left(q^2-1\right) - \left(A_2 - B_2\right)q\right]}{q + B_2 z\left(q^2+1\right) + qB_2^2 z^2},$$

Then it follows that:

$$\frac{1+A_1 z}{1+B_1 z} \prec \Psi^{\beta}_{\tau,q,p}(a,b,c;z) \prec \frac{1+A_2 z}{1+B_2 z},$$

and consequently, the functions $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ represent the sharp subordinant and dominant, respectively, associated with this subordination structure.

Corollary 4.4.22. *Suppose that the conditions stated in references (4.25) and (4.37) are satisfied for the functions $u_1(z) = \frac{1+(1-2\gamma_1)z}{1-z}$ and $u_2(z) = \frac{1+(1-2\gamma_2)z}{1-z}$, respectively, where the parameters fulfill the inequality $0 \leq \gamma_2 \leq \gamma_1 \leq 1$. Let $f \in \mathcal{A}_p$, and assume that the quotient $\frac{\widetilde{V}^{\beta+1}_{\tau,q,p}f(z)}{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}$ belongs to the class $\mathcal{H}[u(0),1] \cap Q$. Furthermore, assume that the function $\Psi^{\beta}_{\tau,q,p}(a,b,c;z)$ defined in (4.26), is univalent in U, and that for constants $a,b,c \in \mathbb{C}$, with $c \neq 0$, the following double subordination holds:*

$$a\frac{1+(1-2\gamma_1)z}{1-z} + b\left(\frac{1+(1-2\gamma_1)z}{1-z}\right)^2 + \frac{cz\left[z(1-2\gamma_1)\left(1-q^2\right) - 2(1-\gamma_1)q\right]}{q - z\left(q^2+1\right) + qz^2} \prec \Psi^{\beta}_{\tau,q,p}(a,b,c;z)$$

$$\prec a\frac{1+(1-2\gamma_2)z}{1-z} + b\left(\frac{1+(1-2\gamma_2)z}{1-z}\right)^2 + \frac{cz\left[z(1-2\gamma_2)\left(1-q^2\right) - 2(1-\gamma_2)q\right]}{q - z\left(q^2+1\right) + qz^2},$$

Thus, it follows that:

$$\frac{1+(1-2\gamma_1)\,z}{1-z} \prec \Psi^{\beta}_{\tau,q,p}\,(a,b,c;z) \prec \frac{1+(1-2\gamma_2)\,z}{1-z},$$

and accordingly, the functions $\frac{1+(1-2\gamma_1)z}{1-z}$ *and* $\frac{1+(1-2\gamma_2)z}{1-z}$ *act as the optimal subordinant and dominant, respectively, within this subordination framework.*

Theorem 4.4.23. *Consider* u *to be an injective and convex mapping within* U*, normalized such that* $u\,(0)=1$*. Suppose that there exist constants,* $m,n\in\mathbb{C}$*, with* $n\neq 0$*, for which the following condition is satisfied:*

$$\mathrm{Re}\left(\frac{m\widetilde{D}_q u\,(z)}{n}\right) > 0. \tag{4.40}$$

Assume that $f\in\mathcal{A}_p$ *and that the function* $\frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}$ *belongs to the class* $\mathcal{H}\,[u\,(0)\,,1]\cap Q$*. Further, let* $\psi^{\beta}_{\tau,q,p}\,(m,n;z)$*, defined as in (4.32), be univalent in* U*. If the subordination condition*

$$mu\,(z)+nz\widetilde{D}_q u\,(z) \prec \psi^{\beta}_{\tau,q,p}\,(m,n;z)\,, \tag{4.41}$$

holds in U*, then the subordination*

$$u\,(z) \prec \frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}, \tag{4.42}$$

follows, and the function u *is the best subordinant.*

Proof. Let $s\,(z)=\frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}$ and $z\in U\backslash\{0\}$, $f\in\mathcal{A}_p$.

We introduce the auxiliary functions:

$$v\,(w)=mw \text{ and } \phi\,(w):=n.$$

Obviously, v is a holomorphic function on $\mathbb{C}$, and ϕ is holomorphic on $\mathbb{C}$ and nonzero and non-vanishing in $\mathbb{C}\backslash\{0\}$.

It can be easily verified that

$$\frac{\widetilde{D}_q v\,(u\,(z))}{\phi\,(u\,(z))} = \frac{m\widetilde{D}_q u\,(z)}{n}.$$

Based on assumption (15), we have

$$\mathrm{Re}\left(\frac{\widetilde{D}_q v\,(u\,(z))}{\phi\,(u\,(z))}\right) = \mathrm{Re}\left(\frac{m\widetilde{D}_q u\,(z)}{n}\right) > 0$$

Now, employing the subordination condition in (4.41), we obtain

$$mu\,(z)+nz\widetilde{D}_q u\,(z) \prec ms\,(z)+nz\widetilde{D}_q s\,(z)\,.$$

An application of (4.4.1) then guarantees the subordination

$$u\,(z) \prec s\,(z) = \frac{\widetilde{V}^{\beta}_{\tau,q,p}f(z)}{z^p}, \quad z\in U,$$

and further establishes that u is the best subordinant. □

Corollary 4.4.24. *Let $u(z) = \frac{1+Az}{1+Bz}$, where $-1 \le B < A \le 1$. Assume that condition (4.40) holds. If $f \in \mathcal{A}_p$ and the function $s(z) = \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$ belongs to the class $\mathcal{H}[u(0), 1] \cap Q$, and if the subordination*

$$m\frac{1+Az}{1+Bz} + nz\frac{A-B}{(1+Bqz)(1+Bq^{-1}z)} \prec \psi^{\beta}_{\tau,q,p}(m,n;z),$$

holds in U, for constants $m,n \in \mathbb{C}$, with $n \neq 0$, where $\psi^{m,n}_{\lambda}(m,n;z)$ is defined in (4.32), then the subordination relation

$$\frac{1+Az}{1+Bz} \prec \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}, \; z \in U,$$

is satisfied, and the function $u(z) = \frac{1+Az}{1+Bz}$ represents the best subordinant.

Proof. The result follows immediately by applying Theorem 4.4.23 with the specific choice $u(z) = \frac{1+Az}{1+Bz}$ and $-1 \le B < A \le 1$, provided that the conditions are satisfied. □

Corollary 4.4.25. *Assume that $u(z) = \frac{1+(1-2\gamma)z}{1-z}, 0 \le \gamma < 1, z \in U$ and assume that condition (4.40) is satisfied. Suppose that the function $f \in \mathcal{A}_p$ and $\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p} \in \mathcal{H}[u(0), 1] \cap Q$. Assume that the subordination*

$$m\frac{1+(1-2\gamma)z}{1-z} + nz\frac{2(1-\gamma)}{(1-qz)(1-q^{-1}z)} \prec \psi^{\beta}_{\tau,q,p}(m,n;z),$$

holds in U for the complex constants m and n, with $n \neq 0$, and where γ is within the range $(0,1]$, where $\psi^{\beta}_{\tau,q,p}(m,n;z)$ corresponds to the definition in (4.32). In that case, the subordination result

$$\frac{1+(1-2\gamma)z}{1-z} \prec \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$$

holds, and the function $\frac{1+(1-2\gamma)z}{1-z}$ is the best possible subordinant.

Proof. The corollary emerges as a special case of Theorem 4.4.23 when choosing $u(z) = \frac{1+(1-2\gamma)z}{1-z}$ for $0 < \gamma \le 1$. □

Corollary 4.4.26. *Let $u(z) = \frac{1+z}{1-z}$, $z \in U$ and assume that condition (4.40) is satisfied. Let $f \in \mathcal{A}_p$, and define the function $s(z) = \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$. Consider also that $s(z) \in \mathcal{H}[u(0), 1] \cap Q$ and that the subordination*

$$m\frac{1+z}{1-z} + nz\frac{2}{(1-qz)(1-q^{-1}z)} \prec \psi^{\beta}_{\tau,q,p}(m,n;z),$$

holds for some constants, $m,n \in \mathbb{C}$, with $n \neq 0$, where the function $\psi^{\beta}_{\tau,q,p}(m,n;z)$ is defined in (4.32). Then the subordination

$$\frac{1+z}{1-z} \prec \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p},$$

holds, and the function $\frac{1+z}{1-z}$ is the best subordinant.

Proof. The conclusion follows directly by applying Theorem 4.4.23 with the specific choice $u(z) = \frac{1+z}{1-z}$, which is a convex univalent function in U, satisfying the requirement $u(0) = 1$. The stated assumptions ensure that all the hypotheses of the theorem are satisfied, and thus the desired result follows. □

Corollary 4.4.27. *Let $u(z) = \frac{1+z}{(1-z)^2}$, $z \in U$ and suppose that condition (4.40) holds. Let $f \in \mathcal{A}_p$ and define the function $s(z) = \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$. Assume additionally that $s(z) \in \mathcal{H}[u(0), 1] \cap Q$ and that the subordination*

$$m\frac{1+z}{(1-z)^2} + nz\frac{(q^{-1}-q)z^2 + (q^{-2}-q^2)z - 3(q^{-1}-q)}{(q-q^{-1})(1-qz)^2(1-q^{-1}z)^2} \prec \psi^{\beta}_{\tau,q,p}(m,n;z),$$

holds for constants $m, n \in \mathbb{C}$, with $n \neq 0$, where $\psi^{\beta}_{\tau,q,p}(m,n;z)$ is defined in (4.32). Then the subordination

$$\frac{1+z}{(1-z)^2} \prec \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p},$$

holds, and the function $\frac{1+z}{(1-z)^2}$ is the best subordinant.

Proof. The result follows as a direct application of Theorem 4.4.23 when taking $u(z) = \frac{1+z}{(1-z)^2}$, a convex univalent function in the unit disk U, normalized by $u(0) = 1$. Given the stated assumptions, all the hypotheses of the theorem are satisfied, thereby ensuring the validity of the conclusion. □

By jointly utilizing the statements of Theorems 4.4.8 and 4.4.23, a unified conclusion is formulated in the form of a double subordination theorem. This finding combines the structures of both lower and upper function dominance, producing a bidirectional inclusion for the analytic function under consideration. Specifically, it demonstrates that the function transformed using the symmetric q-differential operator is bounded above and below by two extremal functions — identified as the best subordinant and best dominant, respectively — each satisfying specific geometric or analytic criteria. This formulation offers a sharp and symmetric enclosure of the transformed function, highlighting the effectiveness of combining subordination and superordination techniques within the structure of symmetric q-calculus. Sandwich-type theorems of this nature are particularly significant in geometric function theory, as they deliver precise information about functional behavior and constraints within complex domains.

Theorem 4.4.28. *Let u_1 and u_2 be holomorphic and one-to-one functions in the unit disk U, where for every $z \in U$, both $u_1(z)$ and $u_2(z)$ do not vanish. Furthermore, assume that the functions $z\widetilde{D}_q u_1(z)$ and $z\widetilde{D}_q u_1(z)$ are both starlike and univalent in U. Suppose that u_1 satisfies condition (4.31) and u_2 satisfies condition (4.40), and let $f \in \mathcal{A}_p$ be such that $\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$ belongs to the class $\mathcal{H}[u(0), 1] \cap Q$. Assume also that the function $\psi^{\beta}_{\tau,q,p}(m,n;z)$ as defined in (4.32) is univalent in U, and that for constants $m, n \in \mathbb{C}$, with $n \neq 0$, the following double subordination holds:*

$$mu_1(z) + nz\widetilde{D}_q u_1(z) \prec \psi^{\beta}_{\tau,q,p}(m,n;z) \prec mu_2(z) + nz\widetilde{D}_q u_2(z),$$

Then it follows that

$$u_1(z) \prec \frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p} \prec u_2(z),$$

and the functions u_1 and u_2 represent, respectively, the best subordinant and the best dominant in this sandwich-type subordination.

Let $u_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $u_2(z) = \frac{1+A_2 z}{1+B_2 z}$, assuming an ordered relation among the parameters, $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$. Under these constraints, the following corollary is valid.

Corollary 4.4.29. *Assume that the conditions specified in (4.31) and (4.40) are satisfied for the functions* $u_1(z) = \frac{1+A_1 z}{1+B_1 z}$ *and* $u_2(z) = \frac{1+A_2 z}{1+B_2 z}$*, respectively, where the parameters are subject to the ordering* $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$*. Let* $f \in \mathcal{A}_p$*, and suppose that the function* $\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$ *belongs to the class* $\mathcal{H}[u(0), 1] \cap Q$*. Assume further that the function* $\psi^{\beta}_{\tau,q,p}(m, n; z)$*, defined in (4.32), is univalent in* U*, and that for constants* $m, n \in \mathbb{C}$*, with* $n \neq 0$*, the following double subordination holds:*

$$m\frac{1+A_1 z}{1+B_1 z} + nz\frac{A_1 - B_1}{(1+B_1 qz)(1+B_1 q^{-1} z)} \prec \psi^{\beta}_{\tau,q,p}(m, n; z)$$

$$\prec m\frac{1+A_2 z}{1+B_2 z} + nz\frac{A_2 - B_2}{(1+B_2 qz)(1+B_2 q^{-1} z)},$$

It can therefore be concluded that

$$\frac{1+A_1 z}{1+B_1 z} \prec \psi^{\beta}_{\tau,q,p}(m, n; z) \prec \frac{1+A_2 z}{1+B_2 z},$$

and consequently, the functions $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *serve, respectively, as the best subordinant and best dominant within this double subordination framework.*

Corollary 4.4.30. *Assume that the conditions specified in (4.31) and (4.40) are satisfied for the functions* $u_1(z) = \frac{1+(1-2\gamma_1)z}{1-z}$ *and* $u_2(z) = \frac{1+(1-2\gamma_2)z}{1-z}$*, respectively, where the parameters fulfill the inequality* $0 \leq \gamma_2 \leq \gamma_1 \leq 1$*. Let* $f \in \mathcal{A}_p$*, and suppose that the function* $\frac{\widetilde{V}^{\beta}_{\tau,q,p} f(z)}{z^p}$ *belongs to the class* $\mathcal{H}[u(0), 1] \cap Q$*. Furthermore, assume that the function* $\psi^{\beta}_{\tau,q,p}(m, n; z)$*, defined in (4.32), is univalent in* U*, and that for constants* $m, n \in \mathbb{C}$*, with* $n \neq 0$*, the following double subordination holds:*

$$m\frac{1+(1-2\gamma_1)z}{1-z} + nz\frac{2(1-\gamma_1)}{(1-qz)(1-q^{-1}z)} \prec \psi^{\beta}_{\tau,q,p}(m, n; z)$$

$$\prec m\frac{1+(1-2\gamma_2)z}{1-z} + nz\frac{2(1-\gamma_2)}{(1-qz)(1-q^{-1}z)},$$

Accordingly, we conclude that

$$\frac{1+(1-2\gamma_1)z}{1-z} \prec \psi^{\beta}_{\tau,q,p}(m, n; z) \prec \frac{1+(1-2\gamma_2)z}{1-z},$$

and therefore, the functions $\frac{1+(1-2\gamma_1)z}{1-z}$ *and* $\frac{1+(1-2\gamma_2)z}{1-z}$ *serve as the optimal subordinant and dominant, respectively, within this subordination framework.*

4.5 Concluding Remarks

In this chapter we developed a unified framework based on a single symmetric q-differential operator that generalizes several classical operators in geometric function theory. This operator, defined through successive iterates and involving the parameters q, α, p and β, allows a flexible treatment of multivalent analytic mappings and naturally extends both the Al-Oboudi and Salagean operators when the quantum parameter tends to one. Using this operator, we introduced three generalized Janowski-type subclasses of multivalent symmetric q-starlike functions. Their structural hierarchy was clarified through a strict chain of inclusions, showing how the geometric restrictions become progressively stronger across the three classes. We also considered a broader Janowski-type family defined by subordination conditions involving the same operator and established coefficient-based criteria ensuring membership in these classes. These results highlight the role of the symmetric q-operator as a unifying tool for capturing fine geometric behavior of analytic functions. A second central theme of the chapter was the development of an extensive subordination and superordination theory associated with the operator. Using symmetric q-versions of classical differential subordination techniques, we derived sharp conditions under which transformed analytic functions are subordinate or superordinate to prescribed univalent functions. In each situation we identified the corresponding best dominant or best subordinant—functions that play an extremal role and describe the optimal geometric bounds for all admissible solutions. By combining both directions, we established sandwich-type theorems that provide two-sided constraints and offer a complete picture of the analytic behavior of the functions under the action of the symmetric q-operator. Overall, this chapter illustrates that the q-deformed operator introduced here serves as a powerful instrument in generalizing classical results, organizing Janowski-type subclasses, and formulating sharp differential inequalities. The methodology and results presented suggest several promising avenues for further research, including extensions to bi-univalent or harmonic mappings, fractional and weighted operator variants, and potential applications in quantum calculus and related areas where symmetric difference structures are of central importance.

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Chapter 5

Classes of p-Valent Meromorphic Functions Defined by a Symmetric q-Differential Operator

Contents

In this chapter (see [1]), we introduce and investigate a new Janowski-type class of meromorphic p-valent functions generated by an iterated symmetric q -differential operator, and we derive sharp analytic and geometric properties for this class. We have defined, in the first chapter, the operator:

$$\widetilde{D}_q f(z) = \begin{cases} \frac{f(qz)-f(q^{-1}z)}{z(q-q^{-1})}, z \neq 0, q \neq 1, z \in U; \\ f'(0), \text{ for } z = 0. \end{cases} \tag{5.1}$$

Also, the normalization condition:

$$f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} a_{j+p} z^{j+p}, \ z \in \mathbb{U}^*. \tag{5.2}$$

is used in this chapter.

5.1 The Symmetric q-Differential Operator and the Associated Janowski-type Meromorphic Classes

In this section, the symmetric q-derivative serves as the basis for introducing a new linear operator acting on $\mathcal{M}_p$. This operator will be instrumental in the definition and analysis of a Janowski-type subclass of meromorphic p-valent functions.

We proceed to define a new differential operator $\widetilde{\mathcal{L}}_{\tau,q} : \mathcal{M}_p \to \mathcal{M}_p$ as follows:

$$\widetilde{\mathcal{L}}_{\tau,q} f(z) = \left(1 + \tau\widetilde{[p]}_q\right) f(z) + \tau z \widetilde{D}_q f(z), \tag{5.3}$$

where $0 < q < 1, \tau \geq 0, z \in \mathbb{U}^*$.

Applying (5.2) and the power series of $\widetilde{D}_q f$ in relation (5.3), we conclude that

$$\widetilde{\mathcal{L}}_{\tau,q} f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} \left(1 + \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right) a_{j+p} z^{j+p}. \tag{5.4}$$

For convenience in what follows, we now introduce the iterates of the operator $\widetilde{\mathcal{L}}_{\tau,q}$ by

$$\begin{aligned} \widetilde{\mathcal{L}}^0_{\tau,q} f(z) &= f(z), \\ \widetilde{\mathcal{L}}^1_{\tau,q} f(z) &= \widetilde{\mathcal{L}}_{\tau,q} f(z) \\ \widetilde{\mathcal{L}}^2_{\tau,q} f(z) &= \widetilde{\mathcal{L}}_{\tau,q}\left(\widetilde{\mathcal{L}}_{\tau,q} f(z)\right) = \frac{1}{z^p} + \sum_{j=1}^{\infty} \left(1 + \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^2 a_{j+p} z^{j+p}. \end{aligned}$$

Analogously, we derive the following result:

$$\widetilde{\mathcal{L}}^\beta_{\tau,q} f(z) = \frac{1}{z^p} + \sum_{j=1}^{\infty} \left(1 + \tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^\beta a_{j+p} z^{j+p}, \ \beta \in \mathbb{N}. \tag{5.5}$$

Remark 5.1.1. *It is worth noting that the operator $\widetilde{\mathcal{L}}_{\tau,q}$, as defined in 5.3, maintains a deep connection with classical differential operators widely used in the theory of harmonic and geometric function theory. In the limit as $q \to 1^-$, the symmetric q-derivative $\widetilde{D}_q$ converges to the ordinary derivative, and hence the operator $\widetilde{\mathcal{L}}_{\tau,q}$, reduces to the classical linear form $\mathcal{L}_\tau f(z) = (1 + \tau p) f(z) + \tau p f'(z)$, which resembles known structures such* *as the Libera operator and certain weighted Salagean-type operators. These operators are fundamental tools in the analysis of starlike, convex, and close-to-convex functions, especially in establishing distortion bounds, growth estimates, and coefficient inequalities. Moreover, when combined with subordination conditions, the operator aligns with the framework of Janowski-type function classes — an established approach for generalizing classical geometric properties via the theory of differential subordinations. This connection reinforces the theoretical foundation of $\widetilde{\mathcal{L}}_{\tau,q}^{\beta}$, positioning it as a natural q-deformation of classical analytic operators, while also enabling the extension of geometric function theory into the realm of symmetric q-calculus.*

Remark 5.1.2. *As $q \to 1^-$, the symmetric q-differential operator defined in (5.3) reduces to the classical differential operator introduced in [4].*

We now use the iterated operator $\widetilde{\mathcal{L}}_{\tau,q}^{\beta}$ to introduce and study a new Janowski-type class of meromorphic p-valent functions. Throughout this section, all results are considered for $z \in \mathbb{U}^*,\ 0 < q < 1,\ 0 \leq A < 1, \tau \geq 0, 0 \leq \alpha < p$ and $\beta \in \mathbb{N}$ unless specified otherwise.

A subclass $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ of $\mathcal{M}_p$ is now introduced, defined through the use of the operator $\widetilde{\mathcal{L}}_{\tau,q}^{\beta}$, as follows.

Definition 5.1.3. *Let f be a meromorphic function in $\mathcal{M}_p$. The function belongs to the subclass $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$, if it satisfies the following condition:*

$$-\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}_{\tau,q}^{\beta}f(z)\right)}{\widetilde{[p]}_q}g(z) \prec \frac{1+z\left[1-A(1+q)\right]}{1-qz} \tag{5.6}$$

for which $g(z) \in \mathcal{SM}_p^*$. The symbol $\prec$ denotes the well-known concept of subordination (see [3]).

Remark 5.1.4. *We say $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ if and only if in the case that $-\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}_{\tau,q}^{\beta}f(z)\right)}{\widetilde{[p]}_q}g(z)$, where $g(z) \in \mathcal{SM}_p^*$, takes all values in the circular domain centered at $\left(1-q-\frac{Aq}{1+q}\right)$ and radius $\frac{A-1}{1-q^2}$.*

In view of standard subordination theory, an equivalent analytic condition for membership in $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ can be easily established and will be frequently used in the proofs of the main results.

An equivalent condition for a function $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ is given below and can be easily established.

$$\left|\frac{\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}_{\tau,q}^{\beta}f(z)\right)}{\widetilde{[p]}_q g(z)}+1}{1-A(1+q)-q\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}_{\tau,q}^{\beta}f(z)\right)}{\widetilde{[p]}_q g(z)}}\right| < 1. \tag{5.7}$$

5.2 Coefficient Conditions, Extremal Functions and Sharp Bounds

In this section, we first derive sufficient conditions that characterize membership in the newly introduced class, $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. Subsequently, we obtain sharp coefficient bounds for functions belonging to this class, thereby contributing to a deeper analytical understanding of their geometric and functional properties.

More precisely we will derive the core analytic conditions that characterize the newly defined subclass $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. These conditions are established using a combination of symmetric q-calculus techniques, subordination principles, and coefficient estimates. The primary goal is to provide a clear and tractable inequality that ensures class membership based on the behavior of the function's coefficients. The first main result, presented below, offers such a condition and serves as the foundation for subsequent corollaries and sharpness results.

Theorem 5.2.1. *Let $f \in \mathcal{M}_p$ be a function with a series representation as given in (5.2). Then f belongs to the class $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ whenever the following inequality is fulfilled:*

$$\sum_{j=1}^{\infty}\left(\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}|a_{j+p}|+\frac{4p\widetilde{[p]}_q}{(q+1)(j+p)}-\frac{2Ap\widetilde{[p]}_q}{j+p}\right)\leq(1-A)\widetilde{[p]}_q. \quad (5.8)$$

This condition is sufficient for membership in the class $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$.

Proof. To validate the sufficiency condition, we aim to verify that the subordination inequality (5.7) is indeed satisfied under the given assumptions.

$$\left|\frac{\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)}+1}{1-A(1+q)-q\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q}g(z)}\right|=\left|\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)+\widetilde{[p]}_q g(z)}{[1-A(1+q)]\widetilde{[p]}_q g(z)-qz\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}\right|.$$

Assume that $g(z)$ may be represented as the following power series:

$$g(z)=\frac{1}{z^p}+\sum_{j=1}^{\infty}b_{j+p}z^{j+p}.$$

To simplify our computations, we define the following compact notation for the expression of the coefficient that appears repeatedly in the summation.

$$\Delta^{\beta}_{j,\tau,p}=\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q.$$

By first applying (5.3), and then making use of (5.1) and (5.4), we deduce — after appropriate simplifications — that the preceding expression is equivalent to:

$$\left|\frac{\sum\limits_{j=1}^{\infty}\Delta^{\beta}_{j,\tau,p}a_{j+p}z^{j+2p}+\sum\limits_{j=1}^{\infty}\widetilde{[p]}_q b_{j+p}z^{j+2p}}{(1-A)(1+q)\widetilde{[p]}_q+[1-A(1+q)]\sum\limits_{j=1}^{\infty}\widetilde{[p]}_q b_{j+p}z^{j+2p}-q\sum\limits_{j=1}^{\infty}\Delta^{\beta}_{j,\tau,p}\widetilde{[j+p]}_q a_{j+p}z^{j+2p}}\right|$$

$$\leq\frac{\sum\limits_{j=1}^{\infty}\Delta^{\beta}_{j,\tau,p}|a_{j+p}|+\widetilde{[p]}_q\sum\limits_{j=1}^{\infty}|b_{j+p}|}{(1-A)(1+q)\widetilde{[p]}_q-[1-A(1+q)]\widetilde{[p]}_q\sum\limits_{j=1}^{\infty}|b_{j+p}|-q\sum\limits_{j=1}^{\infty}\Delta^{\beta}_{j,\tau,p}|a_{j+p}|}.$$

Given that $g(z)\in\mathcal{SM}^{*}_{p}$, it is known from [6] that its coefficients obey a specific inequality that we now apply to simplify the sum.

$$(j+p)|b_{j+p}|=2p. \quad (5.9)$$

Substituting this expression into our earlier inequality and invoking the inequality (5.8), we obtain

$$\left|\frac{\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)}+1}{1-A(1+q)-q\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)}}\right|$$

$$\leq \frac{\sum\limits_{j=1}^{\infty}\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q|a_{j+p}|+2p\widetilde{[p]}_q\sum\limits_{j=1}^{\infty}\frac{1}{j+p}}{(1-A)(1+q)\widetilde{[p]}_q-2[1-A(1+q)]p\widetilde{[p]}_q\sum\limits_{j=1}^{\infty}\frac{1}{j+p}-q\sum\limits_{j=1}^{\infty}\Delta^{\beta}_{j,\tau,p}|a_{j+p}|}.$$

Thus, since the transformed expression remains strictly less than 1, the inequality in (5.7) is satisfied, completing the proof of sufficiency.

In the reverse direction, suppose that $f\in\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ is a function represented by the series expansion given in (5.2). Then, applying (5.7), we deduce that the following relation holds for all $z\in\mathbb{U}^*$:

$$\left|\frac{\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)}+1}{1-A(1+q)-q\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)}}\right|$$

$$=\left|\frac{\sum\limits_{j=1}^{\infty}(1+\tau_{j+p})^{\beta}\widetilde{[j+p]}_q a_{j+p}z^{j+2p}+\sum\limits_{j=1}^{\infty}\widetilde{[p]}_q b_{j+p}z^{j+2p}}{(1-A)(1+q)\widetilde{[p]}_q+[1-A(1+q)]\sum\limits_{j=1}^{\infty}\widetilde{[p]}_q b_{j+p}z^{j+2p}-q\sum\limits_{j=1}^{\infty}\Delta^{\beta}_{j,\tau,p}a_{j+p}z^{j+2p}}\right|.$$

As $|\mathrm{Re}(z)|\leq|z|$, it follows that

$$\mathrm{Re}\left\{\frac{\sum\limits_{j=1}^{\infty}\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q a_{j+p}z^{j+2p}+\sum\limits_{j=1}^{\infty}\widetilde{[p]}_q b_{j+p}z^{j+2p}}{(1-A)(1+q)\widetilde{[p]}_q+[1-A(1+q)]\sum\limits_{j=1}^{\infty}\widetilde{[p]}_q b_{j+p}z^{j+2p}-q\sum\limits_{j=1}^{\infty}\Delta^{\beta}_{j,\tau,p}a_{j+p}z^{j+2p}}\right\}<1. \tag{5.10}$$

Now, let us restrict z to real values such that the expression $\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)}$ remains real-valued. By clearing the denominator in (5.10) and considering the limiting behavior as $z\to 1^-$ on the real line yields (5.8), thereby establishing the desired condition. □

The previous theorem provides a global coefficient condition ensuring that a function belongs to the newly defined class. Next, we comment on the role of the symmetric q-difference operator in this framework.

Remark 5.2.2. *The symmetric q-difference operator used in this study offers several advantages compared to classical q-derivatives and convolution-type operators. Unlike standard forward or backward q-derivatives, the symmetric form uses function values at both qu and $q^{-1}u$, resulting in more stable and balanced behavior near singularities and ensuring better convergence as $q\to 1^-$. The operator exhibits improved rotational invariance, which is beneficial for function classes defined in circular domains such as the punctured unit disk. The operator is embedded in a parametric form $\widetilde{\mathcal{L}}^{\beta}_{\tau,q}$, allowing a tailored analysis through adjustment of τ and β. This gives it broader applicability and adaptability than fixed-kernel convolution operators. As shown in Remark 1.1, the operator* *is reduced smoothly to the classical derivative operator as $q \to 1^-$, ensuring compatibility with well-known results in classical function theory. The operator facilitates the derivation of precise subordination conditions and coefficient bounds, as demonstrated in Theorem 2.1. Such tractable inequalities are less accessible when using traditional convolution techniques. These advantages motivate the use of the symmetric q difference operator to define and analyze the class of functions $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ and justify its theoretical and practical relevance.*

The next result is an immediate specialization of Theorem 6.3.0.1 to the case $p = 1$.

Corollary 5.2.3. *Let $f \in \mathcal{M}$. Then f belongs to the class $\widetilde{\mathcal{KM}}_{\tau,q}(1,\beta,A)$ whenever the following inequality is fulfilled:*

$$\sum_{j=1}^{\infty}\left(\left(1+\tau+\tau\widetilde{[j+1]}_q\right)^{\beta}\widetilde{[j+1]}_q\left|a_{j+1}\right|+\frac{4}{(q+1)(j+1)}-\frac{2A}{j+1}\right)\le 1-A.$$

This condition is sufficient for membership in the class $\widetilde{\mathcal{KM}}_{\tau,q}(1,\beta,A)$.

Example 5.2.4. *For the function*

$$f(z)=\frac{1}{z^p}+\frac{(1-A)\widetilde{[p]}_q-\frac{4p\widetilde{[p]}_q}{(q+1))(p+j)}+\frac{Cp\widetilde{[p]}_q}{p+j}}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+j]}_q\right)^{\beta}\widetilde{[p+j]}_q}z^{p+1},$$

we obtain

$$\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+j]}_q\right)^{\beta}\widetilde{[p+j]}_q\frac{(1-A)\widetilde{[p]}_q-\frac{4p\widetilde{[p]}_q}{(q+1))(p+j)}+\frac{2Ap\widetilde{[p]}_q}{p+j}}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+j]}_q\right)^{\beta}\widetilde{[p+j]}_q}+\frac{4p\widetilde{[p]}_q}{(q+1))(j+p)}$$
$$-\frac{2Ap\widetilde{[p]}_q}{j+p}=(1-A)\widetilde{[p]}_q.$$

Hence, $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$, as well as the inequality (5.8) attains sharpness for this case.

To deepen the understanding of the analytic behavior of functions in the class $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$, we now establish explicit coefficient estimates. These bounds offer direct insight into how the parameters of the symmetric q-operator influence the growth of the function's series expansion. In particular, the next result provides a sharp upper bound for each coefficient a_{j+p}, derived using subordination techniques and classical inequalities. This theorem complements the general inequality in Theorem 5.8 by refining the functional structure of admissible elements within the class.

The following lemma due to Rogosinski plays a key role in establishing our main results.

Lemma 5.2.5. *[5]. Let $\upsilon(z)$ be an analytic function with series representation $\upsilon(z) = 1+\sum\limits_{j=1}^{\infty}\upsilon_j z^j$, and let $\omega(z)$ be a univalent and convex function in $\mathbb{U}$ represented by $\omega(z) = 1+\sum\limits_{j=1}^{\infty}\omega_j z^j$. If $\upsilon(z) \prec \omega(z)$, then it follows that $|\upsilon_j| \le |\omega_1|$, for all $l \in \mathbb{N}^*$.*

Using this lemma in combination with the subordination relation that defines the class, we can now obtain a refined coefficient estimate.

Theorem 5.2.6. *Assume that $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ and has the series representation given by (5.2). Thus*

$$|a_{j+p}| \leq \frac{2p\widetilde{[p]}_q}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q}\left(\frac{1}{j+p}+\sum_{k=1}^{j-1}\frac{(1+q)(1-A)}{k+p}\right).$$

Proof. Assuming that $f \in \mathcal{M}_p$ and lies in the class $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. It must satisfy the condition below:

$$-\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)} \prec \frac{1+[1-A(1+q)]z}{1-qz}.$$

Given that

$$\Phi(z) = -\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^{\beta}_{\tau,q}f(z)\right)}{\widetilde{[p]}_q g(z)}, \tag{5.11}$$

as $\operatorname{Re}\Phi(z)>0$, $\Phi(z)$ can be expressed as

$$\Phi(z) = 1+\sum_{j=1}^{\infty}\phi_j z^j.$$

Thus,

$$\Phi(z) \prec \frac{1+[1-A(1+q)]z}{1-qz}.$$

By performing basic calculations, we arrive at $\frac{1+[1-A(1+q)]z}{1-qz} = 1+(1+q)(1-A)z+...$ and by means of Lemma 5.2.5, we deduce

$$|\phi_j| \leq (1+q)(1-A). \tag{5.12}$$

From (5.5) and (5.11), we obtain

$$1+\sum_{j=1}^{\infty}\phi_j z^j = \frac{\frac{\widetilde{[p]}_q}{z^p}\sum_{j=1}^{\infty}\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q a_{j+p}z^{j+p}}{\widetilde{[p]}_q\left(\frac{1}{z^p}+\sum_{j=1}^{\infty}b_{j+p}z^{j+p}\right)}.$$

After simplification, comparing the corresponding coefficients of z^{j+p} yields

$$-\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q a_{j+p} = \widetilde{[p]}_q b_{j+p}+\widetilde{[p]}_q\sum_{k=1}^{j-1}e_{k+p}\phi_{j-k}.$$

By considering the absolute value of both sides and invoking the triangle inequality along with (5.12), it follows that

$$\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q|a_{j+p}| \leq \widetilde{[p]}_q|b_{j+p}|+(1+q)(1-A)\widetilde{[p]}_q\sum_{k=1}^{j-1}|e_{k+p}|. \tag{5.13}$$

With the aid of (5.13) and (5.9), we arrive at

$$\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q|a_{j+p}| \leq \widetilde{[p]}_q 2p(j+p)^{-1}+(1+q)(1-A)\widetilde{[p]}_q\sum_{k=1}^{j-1}2p(k+p)^{-1}.$$

So

$$|a_{j+p}| \leq \frac{2p\widetilde{[p]}_q}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q\,(j+p)}+\frac{2\,(1+q)\,(1-A)\,p\widetilde{[p]}_q}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q}\sum_{k=1}^{j-1}(k+p)^{-1},$$

and the result follows as required. □

Figures 5.1 and 5.2 offers a visual comparison between the numerical bounds for $|a_{j+p}|$ derived in Theorem 5.2.6. The 2D plot illustrates how the bounds vary with respect to the index l, providing insight into the decay behavior of the coefficients for fixed parameters. In contrast, the 3D plot shows the sensitivity of a specific coefficient bound (for $l = 5$) to changes in the operator parameters q and τ. Together, these plots enhance the interpretation of the theorem, highlighting both the local and parametric effects on the structure of the function class.

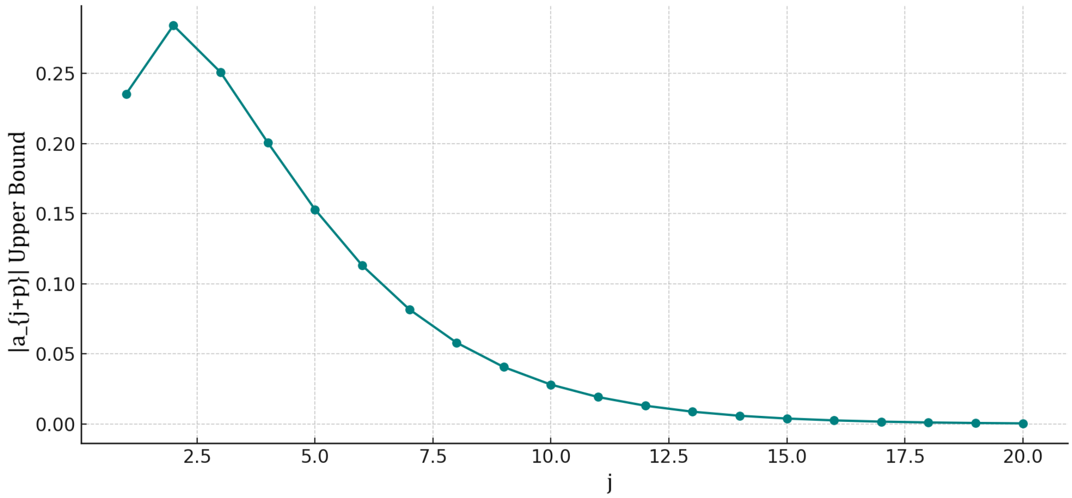

Figure 5.1: Coefficient upper bound (Theorem 5.2.6)

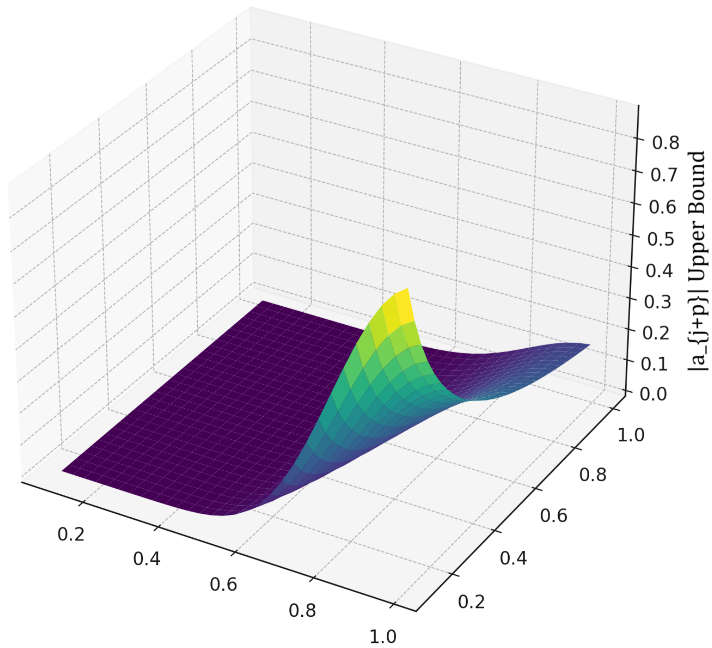

Figure 5.2: Coefficient bound surface for $j = 5$ (Theorem 5.2.6)

As a direct consequence of Theorem 5.2.6, we obtain the following specialization for the univalent case $p = 1$.

Corollary 5.2.7. *If $f \in \widetilde{\mathcal{KM}}_{\tau,q}(1, \beta, A)$ and has the series representation given in (5.2). Then*

$$|a_{j+1}| \leq \frac{2}{\left(1 + \tau + \tau\widetilde{[j+1]}_q\right)^\beta \widetilde{[j+1]}_q} \left(\frac{1}{j+1} + \sum_{k=1}^{j-1} \frac{(1+q)(1-A)}{k+1}\right).$$

5.3 Growth and Distorsion Results for $\widetilde{\mathcal{KM}}_{\tau,q}(p, \beta, A)$

As a natural continuation of the coefficient and distortion estimates, we now focus on bounding the magnitude of the symmetric q -derivative of the transformed function $\widetilde{\mathcal{L}}^\beta_{\tau,q} f(z)$. This result is essential for understanding how the generalized derivative behaves across the punctured disk, particularly under the subordination framework. The next theorem provides sharp upper and lower bounds for the modulus of the operator in terms of the parameters q, τ, β, A and the radial distance r, thereby extending the analytic characterization of the proposed function class.

Theorem 5.3.1. *If $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p, \beta, A)$ and is given by (5.2), assuming $|u| = r$, we arrive at the following estimate:*

$$\begin{aligned} \frac{[1 - (1 - A(1+q)) - r](1-r)^{p+1}\widetilde{[p]}_q}{(1+q-r)r^{p+1}} &\leq \left|\widetilde{D}_q\left(\widetilde{\mathcal{L}}^\beta_{\tau,q} f(z)\right)\right| \\ &\leq \frac{[1 + (1 - A(1+q)) - r](1-r)^{p+1}\widetilde{[p]}_q}{(1-q-r)r^{p+1}}. \end{aligned}$$

Proof. Let us assume that $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p, \beta, A)$. Therefore, we can express

$$-\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^\beta_{\tau,q} f(z)\right)}{\widetilde{[p]}_q g(z)} \prec \frac{1 + (1 - A(1+q))z}{1 - qz}.$$

So, for $|z| = r$, we get

$$\left| -\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}}^\beta_{\tau,q} f(z)\right)}{\widetilde{[p]}_q} g(z) - \frac{1 + [1 - A(1+q)]qr^2}{1 + q^2 r^2} \right| \leq \frac{1 + [1 - A(1+q)]r}{1 - qr}. \tag{5.14}$$

By simplifying, we arrive at

$$\frac{1 - [1 - A(1+q)]r}{1 + qr} \leq \left| -\frac{z\widetilde{D}_q\left(\widetilde{\mathcal{L}^\beta_{\tau,q} f}(z)\right)}{\widetilde{[p]}_q g(z)} \right| \leq \frac{1 + [1 - A(1+q)]r}{1 - qr}. \tag{5.15}$$

Since $g(z) \in \mathcal{SM}^*_p$, it is well known [2] that

$$\frac{(1-r)^{p+1}}{r^p} \leq |g(z)| \leq \frac{(1-r)^{p+1}}{r^p}, \ |z| = r \text{ and } 0 \leq r < 1. \tag{5.16}$$

Making use of (5.15) and (5.16) , the conclusion is established. □

To further explore the geometric behavior of functions in the class $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ we now derive sharp distortion bounds. These estimates describe how the modulus of a function in the class behaves on circles of fixed radius within the punctured unit disk. The following result establishes upper and lower bounds for $|f(z)|$ in terms of the parameters of the symmetric q -operator, thereby providing insight into the radial growth and control of the function across the domain. Such estimates are fundamental in geometric function theory and confirm the analytical stability of the class under symmetric q-deformations.

Theorem 5.3.2. *Let $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. For $|z| = r$, it holds that*

$$|f(z)| \geq \frac{1}{r^p} - \frac{\widetilde{[p]}_q (1-A)}{\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[p+1]}_q\right)^\beta \widetilde{[p+1]}_q} r^p,$$

and

$$|f(z)| \leq \frac{1}{r^p} + \frac{\widetilde{[p]}_q (1-A)}{\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[p+1]}_q\right)^\beta \widetilde{[p+1]}_q} r^p.$$

Proof. Inasmuch that $|z| = r < 1$, we have

$$|f(z)| = \left|\frac{1}{z^p} + \sum_{j=1}^{\infty} a_{j+p} z^{j+p}\right| \leq \frac{1}{|z^p|} + \sum_{j=1}^{\infty} |a_{j+p}| \, |z|^{j+p} = \frac{1}{r^p} + \sum_{j=1}^{\infty} |a_{j+p}| \, r^{j+p}.$$

We get $r^{p+j} < r^p$ and

$$|f(z)| \leq \frac{1}{r^p} + r^p \sum_{j=1}^{\infty} |a_{j+p}| . \tag{5.17}$$

Analogously,

$$|f(z)| \geq \frac{1}{r^p} - r^p \sum_{j=1}^{\infty} |a_{j+p}| . \tag{5.18}$$

Since it has been demonstrated from (5.8) that

$$\sum_{j=1}^{\infty} \left(\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^\beta \widetilde{[j+p]}_q |a_{j+p}| + \frac{4p\widetilde{[p]}_q}{(q+1)(j+p)} - \frac{2Ap\widetilde{[p]}_q}{j+p} \right) \leq (1-A)\widetilde{[p]}_q.$$

However,

$$\begin{aligned} &\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[p+1]}_q\right)^\beta \widetilde{[p+1]}_q \sum_{j=1}^{\infty} |a_{j+p}| \\ &\leq \sum_{j=1}^{\infty} \left(\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[j+p]}_q\right)^\beta \widetilde{[j+p]}_q |a_{j+p}| + \frac{4p\widetilde{[p]}_q}{(q+1)(j+p)} - \frac{2Ap\widetilde{[p]}_q}{j+p} \right). \end{aligned}$$

Therefore,

$$\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[p+1]}_q\right)^\beta \widetilde{[p+1]}_q \sum_{j=1}^{\infty} |a_{j+p}| \leq (1-A)\widetilde{[p]}_q,$$

and is equivalently written as

$$\sum_{j=1}^{\infty} |a_{j+p}| \leq \frac{(1-A)\,\widetilde{[p]}_q}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+1]}_q\right)^{\beta}\,\widetilde{[p+1]}_q}. \tag{5.19}$$

Applying (5.17), (5.18) and (5.19) at this point yields the required result. □

The graphical representations provided in the Figure 5.3, Figure 5.4 and Figure 5.5 illustrate the behavior of the upper and lower distortion bounds for $|f(z)|$, as established in Theorem 5.3.2. These bounds are analyzed with respect to variations in the operator parameters q and τ, while keeping the modulus r fixed. The plots demonstrate how the symmetric q-difference operator influences the growth and contraction of the function within the punctured unit disk. This visual analysis enhances the theoretical results by revealing the sensitivity and stability of the distortion bounds under parameter shifts, offering deeper insight into the geometric properties of the defined function class.

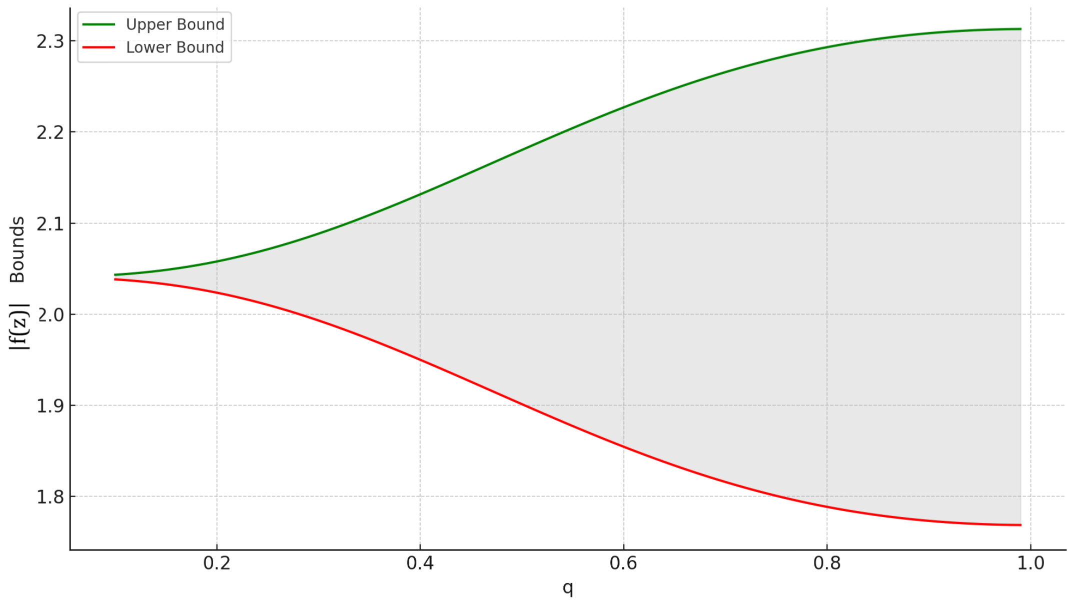

Figure 5.3: representation of upper and lower bounds of $|f(z)|$ as a function of $|u| = r$, (Theorem 5.3.2)

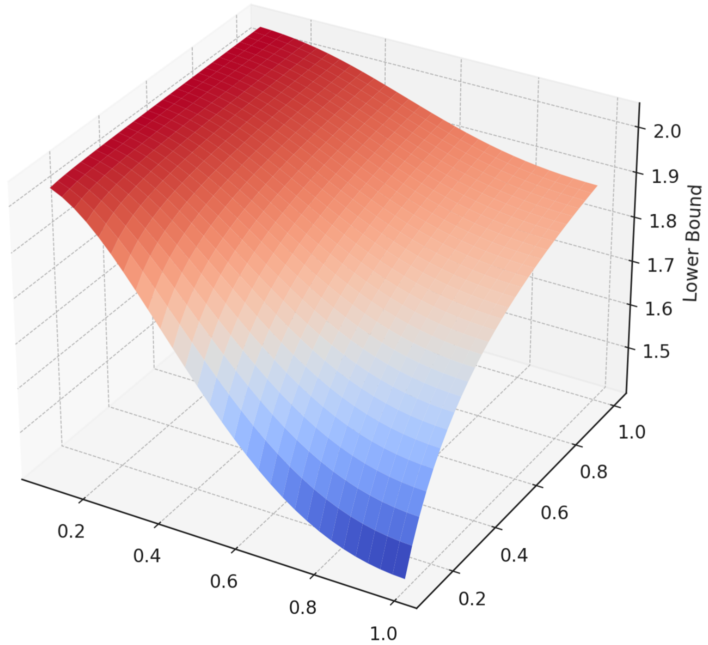

Figure 5.4: Lower bound - Variation of the minimum $|f(z)|$ with respect to q and τ at fixed r=0.7

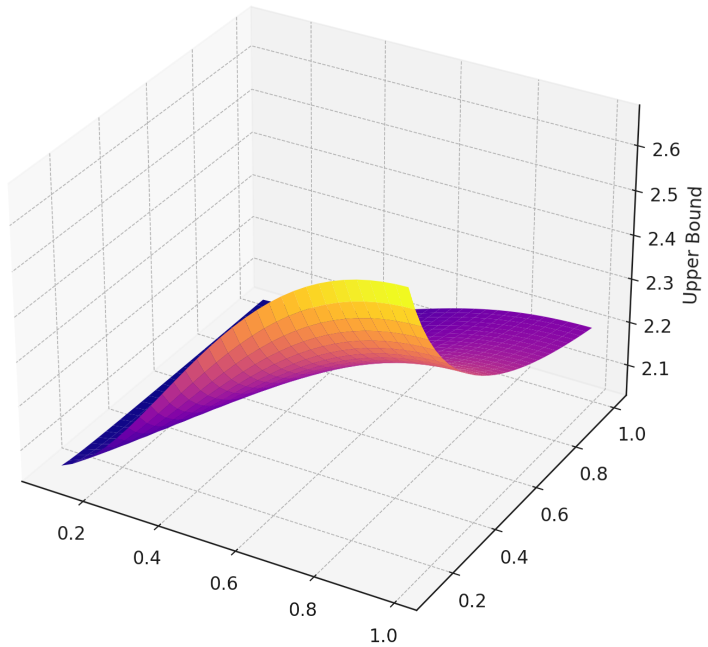

Figure 5.5: Upper bound - Variation of the maximum $|f(z)|$ with respect to q and τ at fixed r=0.7

Corollary 5.3.3. *Let* $f \in \widetilde{\mathcal{KM}}_{\tau,q}(1,\beta,A)$. *For* $|z| = r$, *it holds that*

$$\frac{1}{r} - \frac{1-A}{\left(1+\tau+\tau\widetilde{[2]}_q\right)^{\beta}\widetilde{[2]}_q} r \leq |f(z)| \leq \frac{1}{r} + \frac{1-A}{\left(1+\tau+\tau\widetilde{[2]}_q\right)^{\beta}\widetilde{[2]}_q} r.$$

To complete the analysis of the geometric and analytic behavior of the proposed function class, we now turn to the behavior of higher-order symmetric q-derivatives. This is particularly relevant for assessing the growth and smoothness of functions under multiple iterations of the q-difference operator. Theorem 5.3.4 provides sharp upper and lower bounds for the modulus of the β- th order symmetric q-derivative of $f(z)$, showing how the class remains analytically stable even under repeated operator application. These estimates generalize the earlier results and further demonstrate the robustness of the operator framework.

Theorem 5.3.4. *Assume that $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$ and takes the form described in (5.2). Thus, for every $|z|=r$, we obtain*

$$\left|D_q^{(\beta)} f(z)\right| \leq \frac{\widetilde{[p+\beta-1]}_q!}{r^{p+\beta}\widetilde{[p-1]}_q!} + r^p \frac{\widetilde{[p]}_q (1-A)}{\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[p+1]}_q\right)^{\beta}},$$

and

$$\left|D_q^{(\beta)} f(z)\right| \geq \frac{\widetilde{[p+\beta-1]}_q!}{r^{p+\beta}\widetilde{[p-1]}_q!} - r^p \frac{\widetilde{[p]}_q (1-A)}{\left(1+\tau\widetilde{[p]}_q + \tau\widetilde{[p+1]}_q\right)^{\beta}}.$$

Proof. To obtain the higher-order q-derivative $D_q^{(\beta)} f(z)$, we use the recursive application of the symmetric q-difference operator defined in equation (5.1).

$$D_q^{(2)} f(z) = D_q\left(D_q f(z)\right) = \frac{\widetilde{[p]}_q\widetilde{[p+1]}_q}{z^{p+2}} + \sum_{j=1}^{\infty} \widetilde{[j+p]}_q\widetilde{[j+p-1]}_q a_{j+p} z^{j+p-2}.$$

After observing the pattern from the second derivative $D_q^{(2)} f(z)$, we generalize to the β-th order, obtaining the following representation.

$$\begin{aligned} D_q^{(\beta)} f(z) &= D_q\left(D_q^{(\beta-1)} f(z)\right) \\ &= (-1)^{\beta} \frac{\widetilde{[p]}_q\widetilde{[p+1]}_q \ldots \widetilde{[p+\beta-1]}_q}{z^{p+\beta}} \\ &+ \sum_{j=1}^{\infty} \widetilde{[j+p]}_q\widetilde{[j+p-1]}_q \ldots \widetilde{[j+p-(\beta-1)]}_q a_{j+p} z^{j+p-\beta} \\ &= (-1)^{\beta} \frac{\widetilde{[p+\beta-1]}_q!}{z^{p+\beta}\widetilde{[p-1]}_q!} + \sum_{j=1}^{\infty} \frac{\widetilde{[j+p]}_q!}{\widetilde{[j+p-\beta]}_q!} a_{j+p} z^{j+p-\beta}. \end{aligned}$$

Given that the parameters satisfy $\beta \leq l, l \geq p+1$ and $|z| = r < 1$, one obtains $r^{j-\beta} \leq r^p$. Thus, this leads to

$$\left|D_q^{(\beta)} f(z)\right| \leq \frac{\widetilde{[p+\beta-1]}_q!}{r^{p+\beta}\widetilde{[p-1]}_q!} + r^p \sum_{j=1}^{\infty} \frac{\widetilde{[j+p]}_q!}{\widetilde{[j+p-\beta]}_q!} \left|a_{j+p}\right|. \tag{5.20}$$

Analogously,

$$\left|D_q^{(\beta)} f(z)\right| \geq \frac{\widetilde{[p+\beta-1]}_q!}{r^{p+\beta}\widetilde{[p-1]}_q!} - r^p \sum_{j=1}^{\infty} \frac{\widetilde{[j+p]}_q!}{\widetilde{[j+p-\beta]}_q!} \left|a_{j+p}\right|. \tag{5.21}$$

To bound the series involving $|a_{j+p}|$, we now apply the sufficient condition (5.8), which provides a convenient upper bound in terms of the parameters.

$$\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+1]}_q\right)\sum_{j=1}^{\infty}\widetilde{[j+p]}_q\left|a_{j+p}\right|\leq(1-A)\widetilde{[p]}_q.$$

This leads to

$$\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+1]}_q\right)^{\beta}\sum_{j=1}^{\infty}\widetilde{[j+p]}_q\left|a_{j+p}\right|\leq(1-A)\widetilde{[p]}_q.$$

Thus,

$$\sum_{j=1}^{\infty}\widetilde{[j+p]}_q\left|a_{j+p}\right|\leq\frac{(1-A)\widetilde{[p]}_q}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+1]}_q\right)^{\beta}}.$$

Still, one can easily verify that

$$\sum_{j=1}^{\infty}\frac{\widetilde{[j+p]}_q!}{\widetilde{[j+p-\beta]}_q!}\left|a_{j+p}\right|\leq\sum_{j=1}^{\infty}\widetilde{[j+p]}_q\left|a_{j+p}\right|,$$

giving

$$\sum_{j=1}^{\infty}\frac{\widetilde{[j+p]}_q!}{\widetilde{[j+p-\beta]}_q!}\left|a_{j+p}\right|\leq\frac{(1-A)\widetilde{[p]}_q}{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[p+1]}_q\right)^{\beta}}.$$

Substituting the upper bound into inequalities (5.20) and (5.21) completes the derivation of the desired estimates for the symmetric q-derivatives □

5.4 Radii of Convexity and Starlikeness

In the final part of this chapter, we investigate geometric radii associated with the class $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. In particular, we determine explicit disks in which functions of this class are convex or starlike of a prescribed order. These results extend classical radius theorems to the symmetric q-deformed setting.

The following two theorems establish the convexity and starlikeness radii for functions of the class $\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$.

Theorem 5.4.1. *Consider the case where $f\in\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. Consequently $f\in\mathcal{CM}_p(\alpha)$, within the disk $|z|<p_1$, where*

$$r_1=\left(\frac{p(p-\alpha)\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q}{(p+j)(p+j+\alpha)\widetilde{[p]}_q(1-A)}\right)^{\frac{1}{j+2p}}. \tag{5.22}$$

Proof. Assume that $f\in\widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. In order to establish that $f\in\mathcal{CM}_p(\alpha)$, demonstrating that the following condition holds is sufficient for the proof

$$\left|\frac{zf''(z)+(p+1)f'(z)}{zf''(z)+(1+2\alpha-p)f'(z)}\right|<1.$$

From (5.2) and elementary algebraic manipulation, it follows that

$$\sum_{j=1}^{\infty} \frac{(p+j)(p+j+\alpha)}{p(p-\alpha)} |a_{j+p}| |z|^{j+2p} < 1. \tag{5.23}$$

By applying (5.8), we arrive at:

$$\sum_{j=1}^{\infty} \left(1+\tau\widetilde{[p]_q}+\tau\widetilde{[j+p]_q}\right)^{\beta} \widetilde{[j+p]_q} |a_{j+p}| \leq (1-A)\widetilde{[p]_q}. \tag{5.24}$$

This leads to

$$\sum_{j=1}^{\infty} \frac{\left(1+\tau\widetilde{[p]_q}+\tau\widetilde{[j+p]_q}\right)^{\beta} \widetilde{[j+p]_q}}{\widetilde{[p]_q}(1-A)} |a_{j+p}| \leq 1.$$

Inequality (5.23) holds provided that the following condition is satisfied:

$$\sum_{j=1}^{\infty} \frac{(p+j)(p+j+\alpha)}{p(p-\alpha)} |a_{j+p}| |z|^{j+2p} < \sum_{j=1}^{\infty} \frac{\left(1+\tau\widetilde{[p]_q}+\tau\widetilde{[j+p]_q}\right)^{\beta} \widetilde{[j+p]_q}}{(1-A)\widetilde{[p]_q}} |a_{j+p}|.$$

Hence,

$$|z|^{j+2p} < \frac{p(p-\alpha)\left(1+\tau\widetilde{[p]_q}+\tau\widetilde{[j+p]_q}\right)^{\beta} \widetilde{[j+p]_q}}{(p+j)(p+j+\alpha)\widetilde{[p]_q}(1-A)}.$$

It follows that

$$|z| < \left(\frac{p(p-\alpha)\left(1+\tau\widetilde{[p]_q}+\tau\widetilde{[j+p]_q}\right)^{\beta} \widetilde{[j+p]_q}}{(p+j)(p+j+\alpha)\widetilde{[p]_q}(1-A)}\right)^{\frac{1}{j+2p}} = r_1,$$

and this yields the required result. □

Corollary 5.4.2. *Let $f \in \widetilde{\mathcal{KM}}_{\tau,q}(1,\beta,A)$. Hence $f \in \mathcal{CM}(\alpha)$, in the domain $\{z \in \mathbb{C} : |u| < r_1\}$ under the assumption that*

$$r_1 = \left(\frac{(1-\alpha)\left(1+\tau+\tau\widetilde{[j+1]_q}\right)^{\beta} \widetilde{[j+p]_q}}{(1+j)(1+j+\alpha)(1-A)}\right)^{\frac{1}{j+2}}.$$

The next theorem is devoted to the radius of starlikeness of order α.

Theorem 5.4.3. *Assume that $f \in \widetilde{\mathcal{KM}}_{\tau,q}(p,\beta,A)$. Thus, the function f remains in the class $\mathcal{SM}_p^*(\alpha)$ throughout the subdisk $|z| < r_2$, where*

$$r_2 = \left(\frac{(p-\alpha)\left(1+\tau\widetilde{[p]_q}+\tau\widetilde{[j+p]_q}\right)^{\beta} \widetilde{[j+p]_q}}{(p+j+\alpha)\widetilde{[p]_q}(1-A)}\right)^{\frac{1}{l+2p}}.$$

Proof. It is known that the function f is a member of $\mathcal{SM}_p^*(\alpha)$ if next condition holds:

$$\left|\frac{zf'(z)+pf(z)}{zf'(z)-(p-\alpha)f(z)}\right|<1.$$

From (5.2), after a few simple steps, it can be deduced that

$$\sum_{j=1}^{\infty}\frac{p+j+\alpha}{p-\alpha}\left|a_{j+p}\right|\left|z\right|^{j+2p}<1. \tag{5.25}$$

Making use of (5.24), immediately yields

$$\sum_{j=1}^{\infty}\frac{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q}{\widetilde{[p]}_q\left(1-A\right)}\left|a_{j+p}\right|\leq 1.$$

For inequality (5.25) to hold, the condition reduced to

$$\sum_{j=1}^{\infty}\frac{p+j+\alpha}{p-\alpha}\left|a_{j+p}\right|\left|z\right|^{j+2p}<\sum_{j=1}^{\infty}\frac{\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q}{\left(p+j+\alpha\right)\left(1-A\right)\widetilde{[p]}_q}\left|a_{j+p}\right|. \tag{5.26}$$

This generated

$$\left|z\right|^{j+2p}<\frac{\left(p-\alpha\right)\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q}{\left(p+j+\alpha\right)\left(1-A\right)\widetilde{[p]}_q}.$$

So,

$$\left|z\right|<\left(\frac{\left(p-\alpha\right)\left(1+\tau\widetilde{[p]}_q+\tau\widetilde{[j+p]}_q\right)^{\beta}\widetilde{[j+p]}_q}{\left(p+j+\alpha\right)\left(1-A\right)\widetilde{[p]}_q}\right)^{\frac{1}{j+2p}}=r_2.$$

Therefore we arrive at the desired conclusion . □

Corollary 5.4.4. *Assume that $f\in\widetilde{\mathcal{KM}}_{\tau,q}(1,\beta,A)$. Then f also lies in the class $\mathcal{SM}^*(\alpha)$ within the disk $|z|<r_2$, where*

$$r_2=\left(\frac{\left(1-\alpha\right)\left(1+\tau+\tau\widetilde{[j+1]}_q\right)^{\beta}\widetilde{[j+1]}_q}{\left(1+j+\alpha\right)\left(1-A\right)\widetilde{[p]}_q}\right)^{\frac{1}{j+2}}. \tag{5.27}$$

5.5 Concluding Remarks

The main contribution of this chapter is the introduction of a new Janowski-type class of p-valent functions generated by an iterated symmetric q-differential operator. Unlike existing approaches that rely primarily on classical or one-sided q-derivatives, the operator used here is symmetric, parameter-dependent, and specifically adapted to the geometry of the punctured disk. This yields a genuinely new analytical framework that is not present in earlier studies. A first original aspect of the chapter is the formulation of a sharp and verifiable coefficient condition that characterizes membership in the newly defined class. This criterion, derived through a combination of symmetric q-calculus and subordination theory, unifies and extends several classical results while introducing new behaviours controlled by the operator parameters. The construction of an extremal function showing the optimality of this condition further strengthens its contribution to the literature. A second innovative feature is the development of precise coefficient bounds obtained through a modern application of Rogosinski's lemma within a q-deformed setting. These estimates reveal how the symmetric q-operator influences the growth structure of p-valent meromorphic functions, providing insight not available in earlier works based on standard q-derivatives or convolution operators. Another significant contribution of this chapter is the derivation of sharp growth and distortion estimates for the operator transform and for the functions in the class. These results demonstrate that the symmetric q-differential framework remains stable and analytically well-behaved, even under repeated operator iteration. Such detailed radial control is new in the context of symmetric q-calculus and complements existing results in the analytic and meromorphic settings. Finally, the chapter establishes explicit radii of starlikeness and convexity for the newly introduced class. These radii theorems extend classical geometric criteria to a symmetric q-deformed environment and highlight how the operator parameters govern the preservation of fundamental geometric properties. This represents a substantial advancement over the existing literature, where such radii results are either absent or limited to classical or unidirectional q-operators. In summary, the chapter provides a coherent and genuinely original contribution to the geometric theory of q-deformed multivalent functions. It introduces a new operator, a new class, sharp characterizing inequalities, refined coefficient and distortion results, and explicit geometric radii — all of which open promising perspectives for further developments in symmetric q-calculus and modern geometric function theory.

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